Nagata type statements

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1 Nagata's theorem and conjecture

In the first lecture we will recall Nagata's theorem on the degrees of plane curves with a square number of multiple points at generic positions, and its application to solve Hilbert's 14-th problem. Then we will state Nagata's celebrated conjecture on curves with an arbitrary number of points and we will review the present state of knowledge on it. The main references for this part are [10], [9], [11], [4], [3]. The lecture is organized in three sections:

1.1 Hilbert's 14-th problem

1.2 Ciliberto-Miranda's proof for Nagata's Theorem

1.3 Nagata's conjecture

2 Conjectures on the Mori and nef cones

In the second lecture, we will look at the nef cone and the cone of curves of a blowup of $\mathbb{P}^2$, and their relationship with Nagata's conjecture. We will introduce the notion of Nagata-type ray, and will state some strong forms of Nagata's conjecture in the language of cones. The main references for this part are [8], [9], [5], [3]. The lecture is organized in three sections:

2.1 Cones of $\mathbb{R}$-divisor classes

2.2 Non polyhedrality

2.3 Nagata-type statements for extremal rays

3 Conjectures on valuations

In the third lecture, we generalize the type of conditions imposed to plane curves, from requiring multiple points to imposing certain values with respect to arbitrary rank 1 valuations. This leads to a conjecture analogous to Nagata's for a number of points which is real, rather than a natural number. The main references for this part are [2, Chapter 8], [7], [6]. The lecture is organized in four sections:

3.1 Valuations on projective surfaces

3.2 Cluster of centers of a valuation

3.3 Quasimonomial valuations

3.4 Berkovich topology and Waldschmidt functions

4 Cones of b-divisors

The conjectures presented in lectures 2 and 3 lead to two different ways of approximating the Nagata ray by rational rays of Nagata type, which could conceivably lead to a proof of Nagata's conjecture. If time permits, in this last lecture we will show that both approaches can actually be described in a unified way by using Shokurov's notion of b-divisors in the Zariski-Riemann space [1]. This is work in progress and owes much to discussions with S. Urbinati. The lecture is organized in three sections:

4.1 Zariski-Riemann space and b-divisors

4.2 Relative Zariski decomposition

4.3 Continuity issues

Exercises for the first two lectures

Notations used in these exercises are introduced in the lectures. Some of them can also be found in professor Harbourne's notes.

Lecture 1

1. Let $Z = m_1x_1 + \cdots + m_nx_n$ be a nonzero fat point subscheme of $\mathbb{P}^2$. Show that $1 \leq \hat{\alpha}(I(Z)) \leq \sqrt{\sum m_i^2}$. Hint: look at $[I(kmZ)]_{kd}$ where d/m is rational and close to but bigger than $\sqrt{\sum m_i^2}$ and $k \gg 0$.

2. We say that a collection of $n \geq 3$ points in $\mathbb{P}^2$ is in linear general position if no r of the points are contained in a line. (In particular, the points are all distinct.) Given n points in linear general position, we can perform the standard Cremona transformation on any 3 of the n points; this gives a different collection of n points in $\mathbb{P}^2$. They need not be in linear general position. We say that a collection of n points is in Cremona general position if they are in linear general position and this remains true after any finite sequence of standard Cremona transformations on subsets of 3 points. Show that this corresponds to a countable intersection of open subsets of $(\mathbb{P}^2)^n$.

3. Let δ, m and e be positive integers with $e \geq \delta m$, and let $d = (\delta + 1)m$, $\Delta = e - \delta m$ and $n = 2\delta + 1$. Pick points $p_1, \ldots, p_n \in \mathbb{P}^2$ in Cremona general position. Show that a plane curve of degree d, with multiplicity e at p_1 and multiplicity m at each of $p_2, \ldots, p_r$ can be transformed by Cremona maps into a curve of degree $m - \Delta\delta$ with a point of multiplicity m.

4. Show that, if C is a closed cone in $\mathbb{R}^2$, then it is finitely generated. Give an example of a closed cone in $\mathbb{R}^3$ which is not finitely generated. Is it true that if a semigroup $S \subset \mathbb{Z}^2$ spans a closed cone, then S itself is finitely generated?

Lecture 2

1. Show that $\mathcal{Q}_n \subseteq \overline{\mathrm{NE}}(X_n)$. Hint: use the first exercise of the previous lecture.

2. Let $D_n = (\sqrt{n-1}, 1^n)$ and let $[D_n] \subset \mathbb{R}^{n+1}$ be the corresponding ray. Show that $D_n^2 = -1$, and $D_n \cdot K_n > 0$ for $n \geq 8$. We will denote by $\Delta_n^{\succcurlyeq}$ [resp. $\Delta_n^{\preccurlyeq}$] the set of classes $\xi \in N^1(X)$ such that $\xi \cdot D_n \geq 0$ [resp. $\xi \cdot D_n \leq 0$]. Show that if $n \geq 10$ then all (-1)-rays lie in the cone $\mathcal{D}_n := \mathcal{Q}_n - [D_n]$; and if $n = 10$, all (-1)-rays lie on the boundary of the cone $\mathcal{D}_n$. Hint: [5].

3. Find a non-nef ray in $\partial\mathcal{Q}_{11}^{\succ}$.

4. Prove that the ray $[(7; 3, 2^{10})]$ is of Nagata type. Hint: use the Ciliberto-Miranda method of the first lecture.

References

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