Dynamics on algebraic surfaces, Warsaw 2016

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1 Lecture 1

The (discrete) dynamics of a space starts with iterating its automorphism or a self-map. One of the most important notion associated to the dynamics is the notion of the *topological entropy.* It can be defined as follows. Let X be a compact metric space and $g : X \to X$ be a continuous automorphism (or a surjective self-map). The dynamic of g starts with iterating the map g. For any point x we can consider the orbits $O_x(g) = \{g^n(x), n \geq 0\}$ of the cyclic group (or semi-group) and ask various question, e.g what are periodic orbits, whether the orbits are dense and many others. For example, the *topological entropy* $h(g)$ of g measures how fast the orbits of two points $O_x(g)$ and $O_y(g)$ disperse. The exact definition is a little involved. For any $n > 0$ we let $d_{g,n}$ be the maximum of the distances between points $g^k(x)$ and $g^k(y)$ for $0 \leq k \leq n-1$ and then consider balls $B_{g,n}(x, \epsilon)$ with center at x of radius ϵ with respect to the distance function $d_{g,n}$. Then we let

$$h(g) := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log S(g, \epsilon, n)}{n},$$

where $S(g, \epsilon, n)$ is the number of balls $B_{g,n}(x, \epsilon)$ that are needed to cover X.

Unfortunately, this metric definition is useless in algebraic geometry where X is an algebraic variety and $g \in \mathrm{Aut}(X)$, especially in positive characteristic. Fortunately, we have the following fundamental theorem of Misha Gromov and Yosef Yomdin;

Theorem 1.1. *Let X be a compact Kähler manifold of dimension n and g is a holomorphic automorphism. Then*

$$h(g) = \log \lambda(g^* | H^{2*}(X, \mathbb{C})),$$

where λ is the largest absolute value of eigenvalues of g acting on the even part of the cohomology space.

Now we can take the right-hand-side for the definition of the topological entropy of ab automorphism of a Kähler or an algebraic variety over any field K. If $K \neq \mathbb{C}$, we replace $H^*(X, \mathbb{C})$ with $H^*_{et}(X, \mathbb{Q}_l) \otimes \mathbb{C}$. We will be concerned with the 2-dimensional case, and assume, for simplicity, that $H^1(X, \mathbb{C}) = 0$, so, we obtain that

$$h(g) = \log \lambda(g^*|H^2(X, \mathbb{C})).$$

The number $e^{h(g)} := \lambda(g)$ is also called the *dynamical degree* of f. The reason is the following.

Theorem 1.2. *Fix an ample divisor class h. Then and let*

$$\lambda(g) = \lim_{n\to\infty} (g^*(h), h)^{1/n}.$$

Proof. For any automorphism g, the quantity $|x \cdot h|$ extends to a norm on $H^2(X, \mathbb{C})$. It is known that for any vector of positive norm $\lim_{n\to\infty} ||f^n(x)||^{1/n}$ is equal to the spectral radius of an operator g^*. □

Since g preserves $H^2(X, \mathbb{Z})/Tors \cong \mathbb{Z}^{b_2(X)}$, the dynamical degree must be an algebraic integer.

We use the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{20}(X) \oplus H^{11}(X) \oplus H^{0,2}(X).$$

Recall that $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z})_{\mathbb{C}} := H^2(X, \mathbb{Z}) \otimes \mathbb{C}$. The group $H^2(X, \mathbb{Z})$ contains the subgroup $H^2_{\text{alg}}(X, \mathbb{Z})$ spanned by the fundamental classes of irreducible closed curves on X. It is equal to the image of the Picard group $\text{Pic}(X)$ of isomorphism classes of line bundles on X under the first Chern class map $c_1 : \text{Pic}(X) \to H^2(X, \mathbb{Z})$. The group $H^2_{\text{alg}}(X, \mathbb{Z})$ is known as the *Néron-Severi group* $\text{NS}(X)$ of X and can be defined over any field as group of divisor classes on X modulo algebraic equivalence. It is a finitely generated abelian group, the quotient of this group by the torsion subgroup is denoted by $\text{Num}(X)$. It can be defined as the group of divisor classes modulo numerical equivalence. It is a free abelian group of finite rank $\rho(X)$, called the *Picard number* of X.

We know that eigenvalues of an automorphism g^* of $H^2(X, \mathbb{C})$ are algebraic integers. Recall that by a *theorem of Kronecker* any algebraic integer is either a root of unity of one of its conjugate has absolute value strictly greater than 1.

Note that the complex conjugation sends H^{20} to $H^{0,2}$ and leaves invariant $H^{11}(X)$. This shows that the spaces $H^{20} \oplus H^{02}$ and $H^{1,1}$ are obtained from real vector spaces W_1 and W_2 by extension of scalars. The cup-product equips them with structure of positive definite and and of signature $(1, h^{11}-1)$ inner product, respectively. The group $\mathrm{Aut}(X)$ preserves these structures and hence any $g \in \mathrm{Aut}(X)$ has roots of unity as eigenvalues on W_1 and roots of unity on $W_2/\mathrm{Num}(X)_{\mathbb{R}}$ (which has negative definite signature). Thus we obtain

Proposition 1.3. *The dynamical degree of g is equal to the spectral radius of g acting on the space $\mathrm{Num}(X)_{\mathbb{R}}$. All eigenvalues of g^* acting on the orthogonal complement of $\mathrm{Num}(X)_{\mathbb{R}}$ are roots of unity.*

Let $\mathbb{H}^n$ be the hyperbolic (Lobachevsky space) associated with the vector space $\mathrm{Num}(X)_{\mathbb{R}} \cong \mathbb{R}^n$. Recall that it is the image of $V^+ = \{x \in V : (x,x) > 0\}$ in the projective space $\mathbb{P}(V) = V \setminus \{0\}/\mathbb{R}^*$. There are different models of $\mathbb{H}^n$. For example, we choose a basis in V such that the inner product is given by the standard quadratic form $t_0^2 - t_1^2 - \cdots - t_n^2$. Then in affine coordinates $x_i = t_i/t_0$,

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i^2 < 1\}.$$

Let

$$\overline{\mathbb{H}^n} = \{t_0^2 - t_1^2 - \cdots - t_n^2 \geq 0\}/\mathbb{R}^* = \{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i^2 \leq 1\}$$

The *absolute* of $\mathbb{H}^n$ is the complementary set

$$\overline{\mathbb{H}^n} = \{t_0^2 - t_1^2 - \cdots - t_n^2 = 0\}/\mathbb{R}^* = \{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i^2 = 1\}$$

It is the unit sphere of dimension $n-1$. The distance in this model is $d(x,y) = \log R(a,x,y,b)$, where $R(a,x,y,b)$ is the cross-ration of four points on the line spanned by x, y with the points (a,b) on the absolute. This is the *Klein* or *projective model* of $\mathbb{H}^n$. In another model, a *vector model*, we fix a representative of a point with $(x,x) = 1$. Then the hypersurface $x_0^2 - x_1^2 - \cdots - x_n^2 = 1$ is a two-sheeted hyperboloid. We have to fix one of its connected components by requiring that $x_0 > 0$. Here the distance is defined by the formula $\cosh d(x,y) = (x,y)$. There is also a *Poincaré model* or *conformal model* of $\mathbb{H}^n$. As a set it is the same as the Klein model, but the distance is defined by different formula.

Recall that a *geodesic line* in a Riemannian manifold M is a continuous map $\gamma : \mathbb{R} \to M$ such that $d(\gamma(a), \gamma(b)) = |a - b|$. We also refer to the image of γ as a geodesic (unparameterized) line. For any two distinct points $x, y \in M$ there exists a closed interval $[a, b]$ and a geodesic γ with $\gamma(a) = x, \gamma(b) = y$. It is called the *geodesic segment*. In $\mathbb{H}^n$, such a geodesic is unique and realizes the shortest curve arc from x to y.

For example, the geodesic lines in the Klein model of $\mathbb{H}^2$ are non-empty intersection of lines in $\mathbb{R}^2$ with the interior of the unit disc.

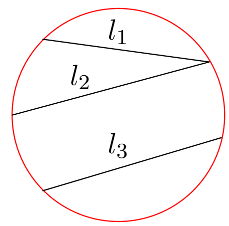

Figure 1: Lines on hyperbolic plane in Klein model

Here the lines l_1 and l_2 are parallel and the lines $l_1(l_2)$ and l_3 are *divergent*.

A *motion* or an *isometry* of a Riemannian manifold is a smooth map of manifolds that preserves the Riemannian metric. In our case the group of motions consists of projective transformations that preserve the projective model of $\mathbb{H}^n$. When we choose the standard coordinates this group is the projective orthogonal group $\mathrm{PO}(1, n)$. We denote it by $\mathrm{Iso}(\mathbb{H}^n)$. Note that $\mathrm{PO}(1, n)$ is isomorphic to the index 2 subgroup $\mathrm{O}(1, n)'$ of $\mathrm{O}(1, n)$ that preserves the connected components of the 2-sheeted hyperboloid $\{x \in \mathbb{R}^{1,n} : q(x) = 1\}$. The group $\mathrm{Iso}(\mathbb{H}^n)$ is a real Lie group, it consists of two connected components. The component of the identity $\mathrm{Iso}(\mathbb{H}^n)^+$ consists of motions preserving an orientation.

Let Γ be a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$. It acts on $\mathbb{H}^n$ totally discontinuously, i.e., for any compact subset K, the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite. In particular, the stabilizer of any point is a finite group. The action extends to the absolute, however here Γ does not act discontinuously. We define the *limit set* $\Lambda(\Gamma)$ to be the complement of the maximal open subset of $\partial\mathbb{H}^n$ where Γ acts discretely. It can be also defined as the closure of the orbit $\Gamma \cdot x$ for any $x \in \mathbb{H}^n$. It is also equal to the closure of the set of fixed points on $\partial\mathbb{H}^n$ of elements of infinite order in Γ. For any $\gamma \in \Gamma$ there are three possible cases:

- γ is *hyperbolic*: there are two distinct fixed points of γ in $\partial\mathbb{H}^n$;
- γ is *parabolic*: there is one fixed point of γ in $\partial\mathbb{H}^n$;

- γ is *elliptic*: γ is of finite order and has a fixed point in $\mathbb{H}^n$;

Let γ be a hyperbolic isometry. Then there exists a hyperbolic plane U in V generated by isotropic vectors u, v with $(u, v) = 1$ such that $\gamma(U) = U$ and $\gamma|U$ is given by the matrix $A(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{\pm 1}$ for some $\lambda > 1$. The fixed points of γ on $\partial\mathbb{H}^n$ are represented by the vectors u and v. The distance between two points $x = [e^t u + e^{-t} v], y = [e^{t'} u + e^{-t'} v]$ represented by vectors with norm 1 on the geodesic $\gamma = \mathbb{P}(U) \cap \mathbb{H}^2 \subset \mathbb{H}^n$ is equal to $\frac{1}{2} \ln |R|$, where R is the cross-ratio of the points $(0, x, y, \infty)$. It is equal to $e^{2(t-t')}$, so we get $d(x, y) = |t - t'|$. So, we see that t is the natural parameter on the geodesic γ, and the isometry γ_λ with $|\lambda| = |t - t'|$ moves x to the point $\gamma(x)$ with $d(x, \gamma(x)) = |\lambda|$. It also shows that one can identify the geodesic γ with the one-parameter subgroup $\{g_t\}_{t \in \mathbb{R}}$ defined by the matrix $e^{A(\lambda)t}$. It is called the *axis* of g. Also, we see that, any geodesic is a geodesic in some 2-dimensional hyperbolic subspace of $\mathbb{H}^n$.

Let γ be a parabolic isometry. Then there exists an isotropic vector u such that $[u] \in \partial\mathbb{H}^n$ is fixed by γ. Then γ leaves invariant $u^\perp$ and acts naturally on $u^\perp/\mathbb{R}u \cong \mathbb{R}^{0,n-1}$. One can choose a hyperbolic plane U as above with a basis u, v and a vector $w \in U^\perp$ such that γ leaves invariant $U \oplus \mathbb{R}w$ and is given in the basis (u, w, v) by the matrix $\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Let us take a one-parameter subgroup generated by a parabolic isometry γ. It consists of transformations γ_t given by the matrices $\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$. It is clear that γ acts in $u^\perp/\mathbb{R}u$ that we can identify with $U^\perp \cong R^{0,n-1}$. If we change the sign of the quadratic form, this becomes the Euclidean vector space.

Now let g be an automorphism of a smooth projective algebraic surface. We call it elliptic, parabolic, hyperbolic if g^* acts on the corresponding hyperbolic space as an isometry of the respective type. Note that an elliptic isometry is of finite order. In general, a finite power of an elliptic automorphism belongs to a connected component of an algebraic group acting on X. The existence of a non-elliptic automorphism of infinite order gives a strong condition on the surface. Running through the classification, we find that X must belong to one of the following types:

- an elliptic surface of Kodaira dimension 1;
- K3 surface;
- Enriques surface;
- abelian surface;

- blow up of ≥ 9 points in the plane.

One can show that a parabolic automorphism preserves an elliptic fibration. In the case of a K3 surface it corresponds to the isotropic vector fixed by g^*. We also see that the orbits of a parabolic automorphism are not dense in X, their closures are elliptic curves.

It follows that the spectral radius of a parabolic or an elliptic automorphism is equal to 1. On the other hand, the spectral radius of a hyperbolic automorphism is $\lambda > 1$. It is an algebraic integer whose all conjugates except $1/\lambda$ are eigenvalues of g^* acting on the orthogonal complement of a hyperbolic plane $U = \mathbb{R}u + \mathbb{R}v$, ane and hence must have absolute value equal to 1. A monic integer polynomial with roots satisfying this property is called a *Salem polynomial*. The real root larger than one of this polynomial is called a *Salem number*.

So, we obtain the following:

Theorem 1.4. *The characteristic polynomial of a hyperbolic automorphism gis the product of a Salem polynomial, and irreducible cyclotomic polynomials. The dynamical degree of g is a Salem number.*

Since one of the roots α satisfies $|\alpha| = 1$, its conjugate $\bar{\alpha} = 1/\alpha$ is also a root. Thus a Salem polynomial is invariant with respect to $x \mapsto x^{-1}$, and hence its coefficients satisfy $a_i = a_{n-i}$. Its degree is the degree of its minimal polynomial.

The smallest known Salem number is the *Lehmer number* λ_{Lem} satisfying the polynomial of degree 10

$$x^{10} + x^9 - (x^7 + \cdots + x^3) + x + 1.$$

It is realized as the dynamical degree of an automorphism of a K3-surface and an automorphism of a rational surface obtained from the projective plane by blowing up 10 points on an irreducible cuspidal cubic curve.

The Lehmer number $\lambda_{\text{Lem}}$is the Coxeter number of the Coxeter group with the Coxeter diagram $T_{2,3,7}$ (the E_8-diagram with two more vertices on the long arm). It is also related to the triangular group $G(2,3,7)$, a subgroup of isometries of the hyperbolic plane generated by reflections into sides of a triangle with angles $\pi/2, \pi/3, \pi/7$. Recall that given a group G with a set of generators S one sets w_n to be the number of elements in G that can be written as words of minimal length n in generators. The growth of (G,S) is a number α such that w_n grows as α^n. The growth number of the triangle group is the Lehmer number.

The most interesting case is a hyperbolic automorphism. It does not exist on elliptic surfaces of Kodaira dimension 1 (because it preserves a unique elliptic fibration on such surface) and on the blow-ups of 9 points (because it preserves the divisor class of the proper transform of a plane cubic through the nine points, hence g^* fixes an integral isotropic vector on the absolute). We saw that it has two eigenvectors on the absolute with real eigenvalues $\lambda, 1/\lambda$. All other eigenvalues are algebraic numbers which are roots of 1.

One can extend the previous notions to the case when a group G acts on X. We assume that G is infinite. We say that it is a *parabolic group* if all its elements are either of finite order or parabolic. We say that G is *hyperbolic* (or *loxodromic*) if it contains at least one non-parabolic element.

Theorem 1.5 (M. Gizatullin)**.** *A group is parabolic if and only if it preserves an elliptic fibration. In this case it contains a free abelian finitely generated subgroup of finite index.*

Recall that an elliptic fibration $f : X \to B$ on a surface X is a morphism whose generic fiber is a smooth curve X_η of genus one over the generic point η of B. If $X_\eta(\eta) \neq \emptyset$, it has a structure of an elliptic curve, i.e. an abelian variety of dimension 1. By a theorem of Mordell-Weil, if $f : X \to B$ does not come from a curve over the ground field K by a base change to η, the group of points $X_\eta(\eta)$ is a free abelian group of finite rank. It is called the Mordell-Weil group and is denoted by $\mathrm{MW}(X_\eta)$. The group of automorphisms of X/B contains this group as a subgroup of finite index. If $X_\eta(\eta) = \emptyset$, we consider the Jacobian variety J_η of the curve X_η. It is an elliptic curve over η and X_η is its torsor, a principal homogeneous space. As soon as X_η acquires a rational point over some extension of η, it becomes isomorphic to J_η over this field. The Mordell-Weil group $\mathrm{MW}(J_\eta)$ acts on X_η by translations, and this defined an injective homomorphism from $\mathrm{MW}(J_\eta)$ to $\mathrm{Aut}(X/B)$ whose image is a subgroup of finite index in $\mathrm{Aut}(X/B)$.

In the theory of hyperbolic spaces a discrete group subgroup of isometries of $\mathbb{H}^n$ that contains an finitely generated abelian subgroup of finite index is called *elementary group* or *virtually abelian.*

The dynamics of a parabolic and hyperbolic automorphism is very different. Since some power of a parabolic automorphism acts as a translation on a general fiber of an elliptic fibration, its periodic orbits are not dense in X. On the other hand, the following very non-trivial result of N. Fakhruddin, A. Hrushevski and Junie Xie is true.

Theorem 1.6. *For any hyperbolic automorphism g of an irreducible projective surface over any algebraically closed field, the set of periodic orbits of g*

is dense in X (in the Zariski topology). For any curve $Z \subset X$, there exists a periodic orbit in $X - Z$.

Let (M, Σ, μ) be a space with a probability measure μ and the set of measurable subsets Σ. An action of a group G on X that preserves the measure is called an *Anosov action* if any the measure of any invariant subset is either 0 or 1. Assume X is a Kahler surface with a Kähler 2-form ω. After appropriate scaling, can the volume form $\omega\wedge\bar{\omega}$ defines a probability measure.

Theorem 1.7 (S. Cantat)**.** *Let X be a Kähler surface. Suppose $G \subset$ $\mathrm{Aut}(X)$ contains a hyperbolic automorphism, then its action on X is Anosov.*

To see examples of hyperbolic groups of automorphisms of a K3 surface, one can use a theorem of V. Nikukin.

Theorem 1.8. *Assume X has two different elliptic fibrations with infinite Mordell-Weil group. Then* $\mathrm{Aut}(X)$ *is hyperbolic.*

2 Lecture 2

Let us consider some examples of computation of dynamical degrees.

Example 2.1. Let $A = E^n$ be the product of n copies of an elliptic curve. The group $\mathrm{GL}_n(\mathbb{Z})$ embeds naturally in $\mathrm{Aut}(A)$ by acting on E^n via sending $(x_1, \ldots, x_n)$ to $(\sum_{i=1}^n [a_{1j}]x_j, \ldots, \sum_{j=1}^n [a_{nj}]x_j)$, where, for any integer m we denote by $[m]$ the homomorphism $E \to E$ of multiplication by m. Let $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ can be identified with $H_1(E, \mathbb{Z})$. This shows that a matrix $M \in \mathrm{GL}_n(\mathbb{Z})$ acts on $H^1(A, \mathbb{C}) = (\Lambda_{\mathbb{C}}^*)^{\oplus n} \cong \mathbb{C}^{2n}$[1] as the direct sum $M \oplus M$. Its eigenvalues are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of M taken with multiplicity 2. Its eigenvalues on $H^k(A, \mathbb{C})$ are the products $\lambda_{i_1} \cdots \lambda_{i_k}$ taken with multiplicity 2. Thus, the spectral radius of M is λ_1^n, where λ_1 is the spectral radius of M.

Assume $n = 2$, i.e. we are dealing with an abelian surface. Then the characteristic polynomial of $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is $t^2 - (a+d)t + 1$, where we assume that $\det M = 1$. The discriminant of the polynomial is $(a + d)^2 - 4$. Thus, if $|a + d| > 2$, we have two roots $\lambda > 1$ and $1/\lambda$, so that the polynomial is a Salem polynomial. If $|a + d| = 2$, we have a unipotent matrix, the transformation is parabolic. If $|a + d| < 2$, then the roots are $\lambda, \bar{\lambda} = 1/\lambda$. The absolute value is equal to 1, and, by Kronecker theorem, they aree roots of unity. Thus the automorphism is elliptic.

[1] In terms of the Hodge decomposition, it is isomorphic to $H^{1,0}(A) \oplus H^{),1}(A)$.

Example 2.2. Assume E is an elliptic curve with complex multiplication by $i=\sqrt{-1}$. Then the group $\mathrm{GL}_2(\mathbb{Z}[i])$ acts on $A=E\times E$. Let σ be an automorphism of order 4 of $E\times E$ that acts by $(x,y)\mapsto(ix,iy)$. Let X be the quotient space $E\times E)/(\sigma)$. The projection $X\to E/(\sigma)=\mathbb{P}^1$ has a general fiber isomorphic to E. It has special fibers over the branch points of $p:E\to E/(\sigma)$. We assume that they are $0,1,\infty\in\mathbb{P}^1$ with $p^{-1}(0)=4x_0, p^{-1}(\infty)=4x_\infty, p^{-1}(1)=2x_1+2x_1'$. If we choose x_0 as the origin in the group law on E, then they are the 2-torsion points in $E=\mathbb{C}/bbZ+\mathbb{Z}\tau$:

$$\epsilon_{0,0}=0,\ \epsilon_{\frac{1}{2},\frac{1}{2}}:=\frac{1}{2}+\frac{1}{2}\tau,\ \epsilon_{\frac{1}{2},0}=\frac{1}{2},\ \epsilon_{0,\frac{1}{2}}=\frac{1}{2}\tau \mod \Lambda.$$

The automorphism σ has fixed points $(\epsilon_{a,b},\epsilon_{c,d})\in\frac{1}{2}\Lambda/\Lambda\times\frac{1}{2}\Lambda/\Lambda$. The stabilizer subgroup of the points $(\epsilon_{a,b},\epsilon_{c,d})$, where $(a,b),(c,d)=(0,0)$ or $(\frac{1}{2},\frac{1}{2})$ is a cyclic group of order 4. The stabilizer subgroup of other fixed points is of order 2. The stabilizer subgroup acts diagonally in the local coordinates at these points. Thus the quotient surface $A/(\sigma)$ has 4 singular points locally isomorphic to the singular points of the affine cone over a Veronese curve of degree 4, the remaining 12 fixed points define 6 ordinary double points on the quotient. The images in the quotient of the fibers over the points $(0,1,\infty)$ are isomorphic to $\mathbb{P}^1$. Let X be a minimal resolution of $A/(\sigma)$. One checks that the projection $f:X\to E/(i)=\mathbb{P}^1$ has 3 singular fibers (considered as effective divisors on X):

$$\begin{aligned} f^{-1}(0) &= 4R_0+2R_1+R_2+R_3,\\ f^{-1}(\infty) &= 4R_0'+2R_1'+R_2'+R_3',\\ f^{-1}(1) &= 2R_0''+R_1''+R_2''+R_3''+R_4'', \end{aligned}$$

where all components are isomorphic to $\mathbb{P}^1$ and the intersection matrices of the components are

$$\begin{bmatrix} -1 & 1 & 1 & 1\\ 1 & -2 & 0 & 0\\ 1 & 0 & -4 & 0\\ 1 & 0 & 0 & -4 \end{bmatrix},\quad \begin{bmatrix} -1 & 1 & 1 & 1\\ 1 & -2 & 0 & 0\\ 1 & 0 & -4 & 0\\ 1 & 0 & 0 & -4 \end{bmatrix},\quad \begin{bmatrix} -2 & 1 & 1 & 1 & 1\\ 1 & -2 & 0 & 0 & 0\\ 1 & 0 & -2 & 0 & 0\\ 1 & 0 & 0 & -2 & 0\\ 1 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Let $X\to\bar{X}$ be the blow-down of the curves R_0,R_0' followed by blowing down the curves R_1,R_1'. The projection $f:X\to\mathbb{P}^1$ defines a relatively minimal elliptic fibration on $\bar{X}$. It Kodaira's notation for its possible singular fibers, we have two fibers of type III (two smooth rational curves tangent at one point) and one fiber of type I_0^*. The elliptic surface $\bar{f}:\bar{X}\to\mathbb{P}^1$ admits a section (e.g. the image of $\{0\}\times E$ on $\bar{X}$). It is known that $K_{\bar{X}}=-[F]$, where $[F]$ is the divisor class of a fiber. The known behavior of the canonical class under a blow-up shows that

$$2K_X = -(R_3 + R_4 + R_3' + R_4').$$

We have

$$|-K_X| = \emptyset,\ |-2K_X| = \{R_3 + R_4 + R_3' + R_4'\},\ K_X^2 = -4.$$

A surface S satisfying the property that $|-K_S| = \emptyset$ but $|-mK_X| \neq \emptyset$ for some $m > 1$ is called a *Coble surface*. They are classified by myself and De-Qi Zhang (Amer. J. Math.).

A relatively minimal rational elliptic surface with a section $\bar{X}$ is obtained by blowing up 9 points in the plane (including infinitely near points) which are the base points of a pencil of plane cubic curves. The pencil is rather special. It is spanned by two reducible plane cubic curves $C_1 + L_1$ and $C_2 + L_2$, where C_i are smooth conics and L_i are lines. They satisfy the following conditions

- C_1 and C_2 are tangent to L_1 (resp. L_2) at points q_1, p_1 (resp. q_2, p_2);
- C_1 is tangent to C_2 at a point p_3 and intersect at two additional points p_4, p_5;
- there exists a line L passing through the point $p_6 = L_1 \cap L_2$ and the points p_4, p_5;
- there exists a line M through the points p_1, p_2, p_3.

The surface $\bar{X}$ is obtained by blowing up the points $p_1, \ldots, p_6$ and three infinitely point to p_1, p_2, p_3. The singular fibers are the proper transforms of the curves $L_1 + C_1, L_2 + C_2$ and $L + 2M$.

Obviously, the group $\mathrm{GL}_2(\mathbb{Z}[i])$ acts on $X = E \times E/(g)$ with kernel of the action equal to scalar matrices. The group $\mathrm{PGL}_2(\mathbb{Z}[i])$ acts faithfully on X and hence on S. The subgroup of unipotent matrices $\begin{bmatrix} 1 & \mathbb{Z}[i] \\ 0 & 1 \end{bmatrix}$ preserves the elliptic fibration and acts as the lift of the Mordell-Weil group of the relatively minimal model of the elliptic fibration. The subgroup of matrices $\begin{bmatrix} 1 & 0 \\ \mathbb{Z}[i] & 1 \end{bmatrix}$ preserve another (isomorphic) fibration corresponding to the projection to the first factor. It is known that these two subgroups generate $\mathrm{SL}_2(\mathbb{Z}[i])$.

Now take an ample divisor h on X that defines an ample divisor class on S. Its pull-back h' on A is an ample class. For any $g \in \mathrm{Aut}(A)$, we have $(g^n(h'), h') = d(\bar{g}^*(h), h)$, where d is the degree of the map $A \to A/(\tau)$ and $\bar{g}$ is the automorphism of X defined by the automorphism g of A. This implies

$$\lim_{n\to\infty}(g^n(h'), h')^{1/n} = \lim_{n\to\infty}(d(\bar{g}^*(h), h))^{1/n}.$$

Thus the dynamical degree of g acting on A coincides with the algebraic degree of $\bar{g}$. Thus we can realize any Salem number of degree 2 on a rational surface.

Example 2.3. Let X be a smooth hypersurface in $(\mathbb{P}^1)^{n+1}$ given by a form a multi-homogeneous form of multi-degree $(2, \ldots, 2)$. It is a Calabi-Yau manifild of dimension n, in particular, a K3 surface if $n = 2$. For any $j = 1, \ldots, n{+}1$, consider the projection to the product of the factors different from the jth factor. It is a map of degree 2 branched along a hypersurface B in $(\mathbb{P}^1)^n$ given by a form of multi-degree $(4, \ldots, 4)$. The double cover $X \to (\mathbb{P}^1)^n$ induces a quadratic extension of the fields of rational functions, and hence a birational involution of X. If $n = 2$, since X is of non-negative Kodaira dimension, it extends to an automorphism of X. If $n > 2$, then it extends to a *pseudo-automorphism* T_i, a birational automorphism whiich is an isomorphism of open subsets with complementary sets of codimension ≥ 2.

Let $\mathrm{UC}(n{+}1)$ be the *universal Coxeter group* in $n{+}1$ generators. It is the free product of $n + 1$ groups of order 2. We have a natural homomorphism $\rho : \mathrm{UC}(n + 1) \to \mathrm{Bir}(X)$ that assigns to a generator γ_i the automorphism T_i.

If we write the equation of X in the form

$$At_1^2 + 2Bt_0t_1 + Ct_0^2 = 0,$$

where A, B, C are forms of degree 2 in projective coordinates of n factors, we obtain the equation of B in the form $B^2 - AC = 0$. In the open subset $t_0 \neq 0, A \neq 0$, we can rewrite the equation in the form $(At + B)^2 + AC - B^2 = 0$, where $t = t_1/t_0$. The involution T_i acts by replacing t with $-\frac{2B}{A} - t$. If $n = 2$, the closed set we are throwing away is given by two equations and consists of isolated points. Since X is a K3 surface, it can be extended to a biregular automorphism. If $n > 3$, we are throwing away a codimension 2 closed subset, so it is only a pseudo-automorphism. In any case T_i acts on the Picard group of X.

Recall that for any *Coxeter group* (W, S) with $\#S = N$ and the basic relations $(s_is_j)^{m_{ij}} = 1$ $(m_{ii} = 1)$ one can consider a natural linear representation (the *Tits representation*) $\rho : W \to \mathrm{O}(V)$, where V is a real vector space equipped with a symmetric bilinear form defined on a basis $(\alpha_1, \ldots, \alpha_n)$ by

$$(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m_{ij}}.$$

The group W acts by reflections, it sends s_i to a reflection

$$\rho(s_i) : v \mapsto v - 2(v, \alpha_i)\alpha_i.$$

It is known that this linear representation is faithful. The group also acts on the dual vector space V^* and preserves the cone (the *Tits cone*)

$$T = \{l \in V^* : l(\alpha_j) \geq 0, j = 1, \ldots, N\}.$$

It is a convex cone and serves as a fundamental domain for the action of W in V^* (see all of this in Humphrey's book "Reflection groups").

We apply this to our case, where $m_{ij} = \infty$, i.e. $(\alpha_i, \alpha_j) = -1, i \neq j, (\alpha_i, \alpha_i) = 1$. The Gram matrix of this basis $((\alpha_i, \alpha_j))$ is non-degenerate and has signature $(1, n)$ if $n \geq 2$. Thus we can use the bilinear form to identify V with its dual V^*. So, the Tits cone can be also considered in V.

The matrix M_i of a reflection s_i in the basis $(\alpha_1, \ldots, \alpha_{n+1})$ is easy to compute. For simplicity of notation, we give it for $i = 1$. We have $s_1(\alpha_1) = -\alpha_1$ and $s_i(\alpha_i) = \alpha_i + 2\alpha_1$.

We know that $\mathrm{Pic}((\mathbb{P}^1)^{n+1}) \cong \mathbb{Z}^{n+1}$ and it is generated by the classes h_i of pull-backs of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the projection to the ith factor. By Lefschetz Theorem, the restriction map $\mathrm{Pic}((\mathbb{P}^1)^{n+1}) \to \mathrm{Pic}(X)$ is an isomorphism if $n \geq 3$ and injective if $n = 2$. If X is general, the latter is also an isomorphism. Assume that this is the case. Let e_i be the images of h_i in $\mathrm{Pic}(X)$.

Let us see how T_i acts on $\mathrm{Pic}(X)$. We assume for the simplicity of notation that $i = 1$. Let $p_1 : X \to (\mathbb{P}^1)^{n-1}$ be the projection to the product of the jth factors except the first one. Then T_1 leaves invariant $p_1^*(\mathrm{Pic}((\mathbb{P}^1)^n)) \cong \mathbb{Z}^{n-1}$. Hence $T_1^* - 1$ is of rank 1, and hence must be a reflection with respect to a vector orthogonal to $p_1^*(\mathrm{Pic}((\mathbb{P}^1)^{n+1}))$. Obviously $h_i^2 = 0$. Since the fundamental class $[X]$ of X in $(\mathbb{P}^1)^{n+1})$ is equal to $2(h_1 + \cdots + h_{n+1})$. Thus

$$(e_{i_1}, e_{i_2}, \ldots, e_{i_k})_X = (h_{i_1}, h_{i_2}, \ldots, h_{i_k}, X)_{(\mathbb{P}^1)^{n+1})}$$

$$= 2 \sum_{j \neq i_1, \ldots, i_k} (h_{i_1}, h_{i_2}, \ldots, h_{i_k}, h_j)_{(\mathbb{P}^1)^{n+1})}.$$

We have $T_1^*(e_i) = e_i, i \neq 1$ and $T_1^*(e_1) + e_1 = \sum_{i\neq 1} a_i e_i$. Intersecting the both sides with $(e_3, \dots, e_n), \dots, (e_2, \dots, e_{n-1})$ we obtain $a = 2$. Thus

$$T_1^*(e_1) = -e_1 + 2(e_2 + \cdots + e_{n+1}), \quad T_1^*(e_i) = e_i, i \neq 1.$$

Comparing this with the matrices for the reflections s_i in the Tits representation of $\mathrm{UC}(n+1)$, we find that it is equal to the matrix of T_i. Thus the action of $\mathrm{UC}(n+1)$ induces the action on $\mathrm{Pic}(X)_{\mathbb{R}}$ isomorphic to the Tits representation.

Assume $n = 2$ (the case $n > 2$ is studied in a paper of S. Cantat and K. Oguiso). We take $g = \rho(s_3 s_2 s_1)$. The matrix of g^* is equal to

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 15 & 10 & -6 \\ 6 & 3 & -2 \\ 2 & 2 & -1 \end{pmatrix}.$$

The characteristic polynomial of A is equal to

$$(x+1)(x^2 - 18x + 1).$$

Thus the dynamical degree of g is equal $9 + 4\sqrt{5}$.

One can extend the notion of the dynamical degree to the case of a birational transformation f of a surface X. The transformation f defines a homomorphism $f_* : \mathrm{NS}(X) \to \mathrm{NS}(X)$ and we define

$$\lambda(f); = \overline{\lim}_{n\to\infty} (f_*^n(h), h)^{1/n},$$

where h is an ample divisor class of X.

In particular, we can define the dynamical degree of a *Cremona transformation* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Recal that such a transformation f is defined by homogeneous polynomials (F_0, F_1, F_2) of some degree $d > 1$ without common factors of degree > 0. They generate a 2-dimensional linear system with two general fibers intersecting with multiplicity one outside of the base scheme $\mathrm{Bs}(f) = V(F_0, F_1, F_2)$. The degree d is the *degree* $\deg(f)$ of f. It is equal to $f^*(e) \cdot e$, where e is the class of a line in $\mathbb{P}^2$. The composition $f \cdot g$ is a Cremona transformation with $\deg(f \circ g) \leq \deg(f) deg(g)$. The equality happens if and only if $\mathrm{Bs}(f^{-1}) \cap \mathrm{Bs}(g) = \emptyset$. A non-trivial result of Diller and Favre says that, replacing f by a conjugate element in $\mathrm{Bir}(\mathbb{P}^2)$, one can find a birational map $\phi : S \to \mathbb{P}^2$ from a rational surface S such that $f' = \phi^{-1} \circ f \circ \phi$

$$(f'^n)_* = (f'_*)^n. \tag{2.1}$$

Thus $f'_* : \mathrm{NS}(S) \to \mathrm{NS}(S)$ defines an action, hence its spectral radius is equal to the dynamical degree and is an algebraic integer. A birational map of a rational surface satisfying property (2.1) is called *algebraically stable*.

Theorem 2.4. *The dynamical degree of a Cremona transformation is an algebraic integer. If it is larger than one, then it is either a Salem number or a* Pisot number *(an algebraic integer > 1 whose all conjugates lie inside of the unit circle).*

It is known the set of Pisot numbers is a closed subset of $\mathbb{R}$. It is contained in the closure of the set of Salem numbers and its minimum is equal to the root of $x^3 - x - 1$ which is approximately equal to 1.324717 (it is called the *plastic* or *padovian* number). The smallest accumulation point is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$. All Pisot numbers between these two numbers are known.

Example 2.5. Let f be a Cremona transformation defined in affine coordinates by $(x, y) \mapsto (y, x + y^d)$. In projective coordinates it is defined by the polynomials $(x_0^d, x_0^{d-1}x_2, x_1x_0^{d-1} + x_2^d)$. It has one base point $(0, 1, 0)$. It is an example, of a *de Jonquières transformation*. Its degree is equal to d. We check that $\deg(f^n) = deg(f)^n$, so the transformation is algebraically stable. Hence the dynamical degree is equal to d. So all positive integers are realized as dynamical degrees of Cremona transformations.

Example 2.6. Consider a monomial birational transformation

$$(x, y) \mapsto (x^{a_{11}}y^{a_{12}}, x^{a_{21}}y^{a_{22}}),$$

where the matrix $A = (a_{ij})$ belongs to $\mathrm{GL}_2(\mathbb{Z})$. One can show that the dynamical degree is equal to the spectral radius of the matrix A.

It follows that a Cremona transformation with dynamical degree > 1 equal to a Pisot number is not conjugate to an automorphism of a rational surface (since in this case it must be a Salem number).

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map and $\lambda(f)$ be its dynamical degree. One has the following alternatives:

- $\lambda(f) = 1$ and $\deg(f^n)$ is bounded. Then f is conjugate to a projective automorphism.
- $\lambda(f) = 1$ and $\deg(f^n)$ is bounded by a linear function. Then f preserves a pencil of rational curves.
- $\lambda(f) = 1$ and $\deg(f^n)$ is bounded by a quadratic function. Then f preserves a pencil of elliptic curves.

- $\lambda(f) > 1$ and $\lambda(f)$ is a Pisot number and f is not conjugate to an automorphism of a rational surface.
- $\lambda(f) > 1$ and $\lambda(f)$ is a Salem number and f is conjugate to an automorphism of a rational surface

It is known that any Salem number that occurs as a spectral radius of an element from the Weyl group W_N of the blow-up of N points in the plane occurs as the dynamical degree of an automorphism of a rational surface. The smallest number among them is the Lehmer number.

One can extend the notion of the dynamical degree of an automorphism of a surface to define the dynamical degree of an infinite group G of an algebraic surface X. One considers its action in the hyperbolic space associated with $\mathrm{Num}(X)_{\mathbb{R}}$. Let Γ be the corresponding discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$. Let $\Lambda(\Gamma) \subset \partial(\mathbb{H}^n)$ be the *limit set* of Γ, the smallest set such that Γ acts totally discontinuously on the complement of this set in the absolute. It is known to be equal to the closure of set of fixed points of elements in Γ in $\partial(\mathbb{H}^n)$. Let δ_Γ be the *Hausdorff dimension* of this set.

Let A be a subset of the Euclidean space $\mathbb{R}^n$. Recall that the *Hausdorff* dimension $\delta(A)$ of A is defined to be the infimum for all $s \geq 0$ for which

$$\mu_s(A) = \inf_{A \subset \cup_j B_j} (\sum_j r(B_j)^s) = 0. \tag{2.2}$$

Here (B_j) is a countable set of open balls of radii $r(B_j)$ which cover A.

For example, if the Lebesque measure of A is equal to 0, then (2.2) holds for $s = n$, hence $\delta(A) \leq n$. The Hausdorff dimension coincides with the Lebesque measure if the latter is positive and finite. A countable set has the Hausdorff measure equal to zero. Also it is known that the topological dimension of A is less than or equal to its Hausdorff dimension.

Fix an ample divisor class h_0. For any ample divisor class h define the function

$$N_{h,\Gamma}(T) = \#\{g \in \Gamma : (g^*(h), h_0) < T\}.$$

Theorem 2.7. *Assume that Γ is not an elementary but geometrically finite subgroup of $O(\mathrm{Num}(S))$)i.e it has a finite polyhedral fundamental domain in $\mathbb{H}^n$). Then*

$$\lim_{T \to \infty} \frac{N_{h,X}(T)}{T^{\delta_\Gamma}} = c_{h,\Gamma},$$

where $c_{h,\Gamma}$ is a constant depending on h, Γ.

So this theorem says that $N_{h,\Gamma}(T)$ grows assymptotically as $c_{h,\Gamma}T^{\delta_\Gamma}$. On the other hand, if Γ is a cyclic group generated by g, the similar number grows as $\log(\lambda(g))\log T$.

The Hausforff dimension is very difficult to compute. If Γ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$ (e.g. when X has no (-2)-curves and $n > 2$, the Hausdorff dimension is equal to $n-1$.

Example 2.8. This example is due to A. Baragar. We consider a nonsingular hypersurface X in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ of type $(2,2,2)$. If X is general, then the Picard lattice coincides with this lattice. We assume that one of the projections $p_{ij} : X \to \mathbb{P}^1\times\mathbb{P}^1$, say p_{12}, contains the whole $\mathbb{P}^1$ as its fiber over some point $q_0 \in \mathbb{P}^1\times\mathbb{P}^1$. All the projections are degree 2 maps. Let $F_i, i = 1,2,3$, be the general fibers of the projections $p_i : X \to \mathbb{P}^1$. Each F_i is an elliptic curve whose image under the map p_{jk} is a divisor of type $(2,2)$. Let f_1, f_2, f_3, r be the classes of the curves F_1, F_2, F_3, R. We assume that X is general with these properties so that $\mathrm{Pic}(X)$ is generated by these classes. The Gram matrix of this basis is equal to

$$\begin{pmatrix} 0 & 2 & 2 & 0\\ 2 & 0 & 2 & 0\\ 2 & 2 & 0 & 1\\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

It is easy to see that

$$\mathrm{Pic}(X) \cong \mathsf{U} \oplus \left(\begin{smallmatrix} -4 & 2\\ -2 & -8 \end{smallmatrix}\right).$$

According to Vinberg's classification of 2-reflective hyperbolic lattices of rank 4, the Picard lattice is not 2-reflective. Hence the image of the group $\mathrm{Aut}(X)$ in $\mathrm{O}(\mathrm{Pic}(X))$ is of infinite index.

Let Φ_{ij} be the automorphisms of X defined by the deck transformations of the projections p_{ij}. Let Φ_4' be defined as the transformation Φ_3 in the previous example with respect to the elliptic pencil $|F_3|$ with section R. The transformation Φ_{12}^* leaves the vectors f_1, f_2, r invariant, and transforms f_3 to $2f_1 + 2f_2 - f_3 - r$. Thus Φ_{12}^* is the reflection with respect to the vector $\alpha_1 = -2f_1 - 2f_2 + 2f_3 + r$.

The transformation Φ_{13}^* leaves f_1, f_3 invariant and transforms r in $r' = f_1 - r$. It also transforms f_2 to some vector $f_2' = af_1 + bf_2 + cf_3 + dr$. Computing $(f_2', f_1) = (f_2, f_1), (f_2', f_3) = (f_2, f_3), (f_2', r) = (f_2, f_1 - r)$, we find that $f_2' = 2f_1 - f_2 + 2f_3$. Similarly, we find that $\Phi_{23}^*(f_2) = f_2, \Phi_{23}^*(f_3) = f_3, \Phi_{23}^*(r) = f_2 - r$ and $\Phi_{23}^*(f_2) = -f_1 + 2f_2 + 2f_3$.

It follows from the definition of a group law on an elliptic curve that

$$\Phi_4'^*(f_3) = f_3,\ \Phi_4'^*(r) = r,\ \Phi_4'^*(f_i) = -f_i + 8f_3 + 4r,\ i = 1,2$$

Consider the transformations

$$\Phi_1 = \Phi_{12},\ \Phi_2 = \Phi_{13} \circ \Phi_{12} \circ \Phi_{13},\ \Phi_3 = \Phi_{23} \circ \Phi_{12} \circ \Phi_{23},\ \Phi_4 = \Phi_4' \circ \Phi_{12} \circ \Phi_4'.$$

These transformations act on $\mathrm{Pic}(X)$ as the reflections with respect to the vectors

$$\begin{aligned}
\alpha_1 &= -2f_1 - 2f_2 + 2f_3 + r, \\
\alpha_2 &= \Phi_{13}^*(\alpha_1) = -5f_1 + 2f_2 - 2f_3 - r, \\
\alpha_3 &= \Phi_{23}^*(\alpha_1) = 2f_1 - 5f_2 - 2f_3 - r, \\
\alpha_4 &= \Phi_4(\alpha_1) = 2f_1 + 2f_2 - 30f_3 - 15r.
\end{aligned}$$

The Gram matrix of these four vectors is equal to

$$\begin{pmatrix} -14 & 14 & 14 & 210 \\ 14 & -14 & 84 & 182 \\ 14 & 84 & -14 & 182 \\ 210 & 182 & 182 & -14 \end{pmatrix} = -14 \begin{pmatrix} 1 & -1 & -1 & -15 \\ -1 & 1 & -6 & -13 \\ -1 & -6 & 1 & -13 \\ -15 & -13 & -13 & 1 \end{pmatrix}$$

Let P be the Coxeter polytope defined by this matrix. The Coxeter group Γ_P is generated by the reflections $\Phi_i^*, i = 1,2,3,4$.

Baragar proves that the automorphisms Φ_{ij} and Φ_4' generate a subgroup Γ of $\mathrm{Aut}(X)$ of finite index. His computer experiments suggest that

$$1.286 < \delta_\Gamma < 1.306.$$

Our reflection group Γ_P generated by $\Phi_1, \ldots, \Phi_4$ is of infinite index in Γ. So, we obtain

$$\delta_{\Gamma_P} < 1.306.$$