Calabi-Yau threefolds and sheaf counting. Exercises II

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1. Recall that if $Z \subset X$ is a proper subscheme in a quasiprojective scheme, then the tangent space to the Hilbert scheme of X at $[Z]$ is $\mathrm{Hom}_Z(I_Z/I_Z^2, \mathcal{O}_Z)$.

   (a) Let $I_Z = m_0^2$ be the square of the maximal ideal of the origin in the affine plane $\mathbb{A}^2$. Show that the Hilbert scheme of points of $\mathbb{A}^2$ is smooth at the point $[Z]$.

   (b) Let $I_W = m_0^2$ be the square of the maximal ideal of the origin in affine space $\mathbb{A}^3$. Show that the Hilbert scheme of points of $\mathbb{A}^3$ is singular at the point $[W]$.

2. Fix $d \geq 2$. Consider a linear subspace I of $\mathbb{C}[\mathbb{A}^d]$ with the property that
$$m_0^r \supset I \supset m_0^{r+1}$$
for some positive integer r, with m_0 the maximal ideal of the origin in $\mathbb{A}^d$. Show that I is in fact an ideal in $\mathbb{C}[\mathbb{A}^d]$. Conclude that the dimension of the Hilbert scheme of points of $\mathbb{A}^d$ grows at least like a constant times $n^{2-2/d}$ as $n \to \infty$.

3. Let G be a finite group acting on affine space $\mathbb{A}^n$, and let $\mathcal{A} = \mathbb{C}[\mathbb{A}^n]\#\mathbb{C}G$ be the corresponding smash product algebra. Show that the centre of the algebra $\mathcal{A}$ is isomorphic to the invariant algebra $\mathbb{C}[\mathbb{A}^n]^G$.

4. Let $\mathcal{P}$ be the set of ordinary partitions, i.e. strings of positive integers
$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1).$$
For a partition λ, let $w(\lambda) = \sum \lambda_i$ be the weight of λ, and $l(\lambda) = k$ be the length of λ (the number of nonzero parts). Show that
$$1 + \sum_{\lambda \in \mathcal{P}} q^{w(\lambda)} t^{l(\lambda)} = \prod_{m \geq 1} (1 - q^m t)^{-1}.$$

5. Check explicitly that the first few terms of the q-expansion of the MacMahon function
$$M(q) = \prod_{m \geq 0} (1 - q^m)^{-m}$$
enumerate three-dimensional partitions with a small number of boxes.

6. Let α be a 3-dimensional partition, represented as a subset of $\mathbb{N}^3$, the positive integer octant. Slice this 3-dimensional partition by planes $\{x - y = c\}$ for integer constants c, to get (2-dimensional pictures of) a set of ordinary partitions $\{\alpha_c \colon c \in \mathbb{Z}\}$. Find the combinatorial condition that needs to be satisfied for a set of ordinary partitions to be slices of a single 3-dimensional partition. (This combinatorial condition plays an important role in Okounkov–Reshetikhin's proof of the MacMahon formula and its generalizations.)