Calabi-Yau threefolds and sheaf counting. Exercises I

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1. Consider weighted projective space $\mathbb{P}^n[a_0, \ldots, a_n]$ with well-formed weights. Show that the $\mathbb{C}^*$-action defining it has fixed point sets along loci corresponding to maximal subsets of $I \subset \{0, \ldots, n\}$ such that
$$d = HCF(a_j : j \in I)$$
is some fixed integer greater than 1, with corresponding stabilizer group $\mathbb{Z}/d\mathbb{Z}$. Describe the geometry of the fixed point sets of the WPS's $\mathbb{P}[1, 1, 2]$ and $\mathbb{P}[1, 1, 2, 4]$.

2. Let S be a finitely generated, non-negatively graded $\mathbb{C}$-algebra $S = \oplus_{n\geq 0} S_n$ with $S_0 \cong \mathbb{C}$. Recall that $\mathrm{Proj} S \cong \mathrm{Proj} S_{(m)}$ where $S_{(m)} = \oplus_{n\geq 0} S_{mn}$.
   (a) Show that $\mathbb{P}[ka_0, \ldots, ka_n] \cong \mathbb{P}[a_0, \ldots, a_n]$ for any positive integer k.
   (b) Show that $\mathbb{P}[a_0, ka_1, \ldots, ka_n] \cong \mathbb{P}[a_0, \ldots a_n]$ for any positive integer k. [This is the reason why we can restrict to well-formed WPSs!]
   (c) Study what happens to a degree d hypersurface in the WPS's on the left hand side under these isomorphisms.

3. (a) Find all complete intersection Calabi–Yau 3-folds (defined by equations of degree at least 2) in ordinary projective spaces.
   (b) Find some nonsingular Calabi–Yau 3-fold hypersurfaces in weighted projective 4-spaces. [Use the formula that the canonical class of $\mathbb{P}^n[a_0, \ldots, a_4]$ is $\mathcal{O}(-\sum a_i)$; you can also assume that for nice values of d, the adjunction formula holds for a degree d hypersurface. Use also question 1.]

4. (a) Consider the variety $\bar{X} = X_{12} \subset \mathbb{P}[1, 1, 2, 2, 6]$. Prove that the birational map $\mathbb{P}[1, 1, 2, 2, 6] \dashrightarrow \mathbb{P}^1$ defined by projection on the first two coordinates defines a rational K3 fibration, describing its general fibre. [Use question 2 above! In fact it is possible to prove that this birational fibration on $\bar{X}$ becomes a genuine K3 fibration, a morphism, on its minimal resolution X.]
   (b) Find a similar example of a (singular) Calabi–Yau 3-fold which has a (birational) K3 fibration with general fibre a smooth quartic in $\mathbb{P}^3$.

5. Let the group $G = \mathbb{Z}/3\mathbb{Z}$ act on $\mathbb{A}^3$ by weights $(1, 1, 1)$. Describe the quotient $\bar{X} = \mathbb{A}^3/G$ as the affine cone over the projective plane $\mathbb{P}^2$ in a specific projective embedding. Deduce that $\bar{X}$ has a resolution of singularities $X \to \bar{X}$ which is the total space of the bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$, with exceptional set the 0-section. Deduce from question 9 below that X is a (nonprojective) Calabi–Yau 3-fold.

6. Let E be the elliptic curve which admits an automorphism of order 3. The group $G = \mathbb{Z}/3\mathbb{Z}$ acts on $E\times E\times E$ diagonally. Show that the quotient contains 27 singular points all of which are quotient singulatities of the form desribed in question 5. Thus the blowup X of $(E\times E\times E)/G$ is a smooth (weak) Calabi-Yau 3-fold which contains 27 planes.

7. Let X be the hypersurface

$$X = \left\{ y_1 \sum_i x_i^4 + y_2 \prod_i x_i = 0 \right\} \subset \mathbb{P}^3 \times \mathbb{P}^1.$$

Here x_i are homogeneous coordinates on $\mathbb{P}^3$ and y_j are those on $\mathbb{P}^1$. Show that X is a smooth Calabi-Yau 3-fold that contains four $\mathbb{P}^2$'s.

8. Let

$$\bar{X} = \{xy = zy\} \subset \mathbb{A}^4$$

be the threefold node (or ordinary double point). Show that $\bar{X}$ has a resolution of singularities X which is the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1,-1)$ and is thus a (quasiprojective) Calabi–Yau threefold by question 9.

9. Let Y be the total space of a vector bundle $\mathcal{N}$ on a smooth projective variety X. Show that the canonical bundle of Y is $K_Y \cong \det(\mathcal{N}^\vee) \otimes K_X$. Hence deduce that Y is a (weak, quasi-projective) Calabi–Yau variety if and only if $\mathcal{N}$ has canonical determinant.