LECTURES ON ASYMPTOTICS OF LINEAR SYSTEMS WITH CONNECTIONS TO LINE ARRANGEMENTS

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Contents

Abstract. The main focus of these lectures is recent work on linear systems in which line arrangements play a role, including problems such as semi-effectivity, containment problems of symbolic powers of homogeneous ideals in their powers, bounded negativity, and a new perspective on the SHGH Conjecture. Along the way we will be concerned with asymptotic invariants such as Waldschmidt constants, resurgences and H-constants.

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1. Line arrangements, semi-effectivity and Waldschmidt constants

1.1. **Line arrangements.** We will always take K to be an algebraically closed field. A line arrangement over K is a finite list $L_1, \dots, L_s \subset \mathbb{P}^2_K$, $s > 1$, of distinct lines in the projective plane and their crossing points (i.e., the points of intersections of the lines). Line arrangements have been coming up in a range of topics of recent research interest that we will be looking at. A useful notation is t_k, for $k \geq 2$, for the number of points lying on exactly k lines.

Exercise 1.1.1. Consider a line arrangement $L_1, \dots, L_d \subset \mathbb{P}^2_K$. Let s be the number of crossing points.

(a) Show that the number of crossing points is $s = t_2 + \dots + t_d$.
(b) Show that $\binom{d}{2} = \sum_k t_k \binom{k}{2}$.
(c) Show that $0 \leq t_d \leq 1$, and that $t_k = 0$ for all $k < d$ if and only if $t_d = 1$.
(d) Show that $d^2 - \sum_k t_k k^2 = d - \sum_k t_k k$.
(e) If the lines do not all go through a single point, show $s \geq d$. (Hint: This is a weak form of the de Bruijn-Erdős theorem in incidence geometry. See [1] for a combinatorial proof. Here is a sketch for an algebraic geometric proof. Blow up the crossing points. Look at the classes of the proper transforms of the lines. Show that they are linearly independent in the divisor class group of the blow up and span a negative definite subspace.)

An interesting property that a line arrangement can have is the $t_2 = 0$ property; i.e., that whenever two of the lines L_i cross, there is at least one other line that also goes through that crossing point. An easy such example is the case of $s \geq 3$ concurrent lines (i.e., $s \geq 3$ lines through a point p). Over the reals, these are the only line arrangements with $t_2 = 0$, due to the following result [3]:

Theorem 1.1.2. *Given a real line arrangement of s lines with $t_s = 0$ (i.e., the lines are not concurrent), we have*

$$t_2 \geq 3 + \sum_{k>2} t_k(k-3).$$

If $\mathrm{char}(K) = p > 0$, there are many examples of line arrangements with $t_2 = 0$.

Exercise 1.1.3. Assume $\mathrm{char}(K) = p > 0$. Consider the arrangement of all lines defined over the finite field $F_q \subset K$ of order q. Show that there are $q^2 + q + 1$ lines and $q^2 + q + 1$ crossing points, that $t_k = 0$ except for $t_{q+1} = q^2 + q + 1$, that each line contains $q + 1$ of the points and that each point is on $q + 1$ of the lines.

Over $K = \mathbb{C}$ only four kinds of line arrangements seem to be known with $t_2 = 0$. Here is the list.

- Any set of $s \geq 3$ concurrent lines.
- The Fermat arrangement of $3n$ lines for $n \geq 3$: The lines of this arrangement are defined by the factors of $(x^n - y^n)(x^n - z^n)(y^n - z^n)$, shown for $n = 3$ in Figure 1. Each line contains $n+1$ of the points, and we have $t_k = 0$ except for $t_3 = n^2$ and $t_n = 3$ when $n > 3$ or $t_3 = 12$ when $n = 3$.
- The Klein arrangement of 21 lines [2]: here $t_k = 0$ except for $t_4 = 21$ and $t_3 = 28$. For this arrangement, each line contains 4 points where 3 lines cross and 4 points where 3 lines cross.
- The Wiman arrangement of 45 lines [4]: here $t_k = 0$ except for $t_5 = 36$, $t_4 = 45$ and $t_3 = 120$. For this arrangement, each line contains 4 points where 5 lines cross, 4 points where 4 lines cross and 8 points where 3 lines cross.

Exercise 1.1.4. Show that $t_2 \neq 0$ for the Fermat arrangement if and only if $n = 2$. Note that the Fermat arrangement is defined over the reals for $n = 1, 2$; draw it in those cases.

Open Problem 1.1.5. *Show either that there are other complex line arrangements with $t_2 = 0$, or that the four types listed above are the only ones.*

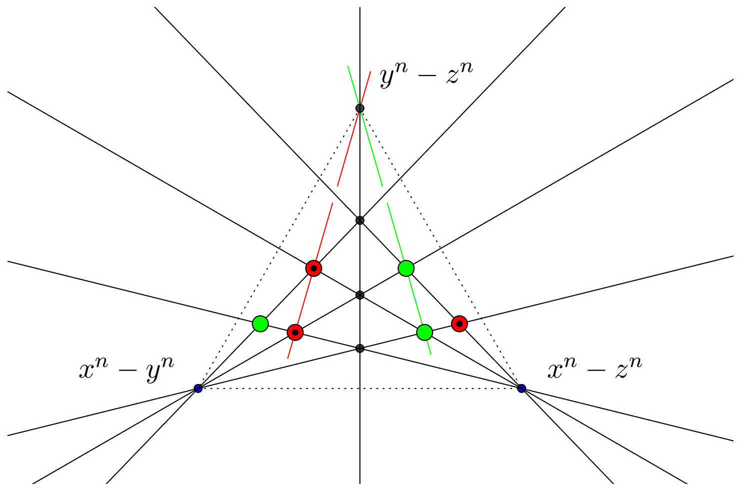

FIGURE 1. The Fermat arrangement of $3n$ complex lines and their $n^2 + 3$ points of intersection (indicated by filled in circles) for $n = 3$. (The coordinate axes are represented by dotted lines. At each coordinate vertex there occur n of the $3n$ lines, defined by the forms shown; the $n^2 + 3$ points consist of a complete intersection of n^2 points plus the 3 coordinate vertices. This arrangement does not exist over the reals: one must regard the open green circles as representing collinear points, and likewise the dotted red circles as representing collinear points.)

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1.2. **Semi-effectivity.**

Definition 1.2.1. Let C be a plane curve defined as a scheme by a nonzero homogeneous polynomial $F \in R = K[x,y,z] = K[\mathbb{P}^2]$. Then the multiplicity of C or F at $p \in \mathbb{P}^2$, denoted $\mathrm{mult}_p(C)$ or $\mathrm{mult}_p(F)$, is the largest m such that $F \in I(p)^m$.

Example 1.2.2. The multiplicity of $F = x^3y^4 + x^5z^2$ at $p = [0:0:1]$ is 5.

Definition 1.2.3. Given two curves C and D defined by nonconstant forms F and G with no common factors, we define the intersection multiplicity of C and D at p by $I_p(C,D) = \dim_K[R/J]_t$ for $t \gg 0$, where $J = I(p)^m + (F,G)$ for any $m \geq \deg(F)\deg(G)$.

Theorem 1.2.4 (Bezout's Theorem)**.** *Let C and D be curves defined by nonconstant forms F and G with no common factors. Then $\sum_p I_p(C,D) = \deg(C)\deg(D)$. Moreover, $I_p(C,D) \geq \mathrm{mult}_p(C)\,\mathrm{mult}_p(D)$ for each point $p \in \mathbb{P}^2$.*

Corollary 1.2.5. *Let C and D be plane curves defined by nonconstant forms F and G. Let $S \subset \mathbb{P}^2$ be a finite set of points. If $\sum_{p\in S} \mathrm{mult}_p(C)\,\mathrm{mult}_p(D) > \deg(C)\deg(D)$, then C and D have a common component (i.e., F and G have a common factor of positive degree).*

Consider distinct points $p_1,\dots,p_s \in \mathbb{P}^N$. Let $\pi : X \to \mathbb{P}^N$ be the blow up of the points. Let L be the pullback of a general hyperplane and let E_i be the inverse image of p_i. Then the divisor class group $\mathrm{Cl}(X)$ is free abelian with basis given by the divisor classes $[L],[E_1],\dots,[E_s]$. When $N = 2$, this is an orthogonal basis for the intersection form on $\mathrm{Cl}(X)$, with $-L^2 = E_1^2 = \cdots = E_s^2 = -1$ and we have $-K_X = 3L - E_1 - \cdots - E_s$.

Given $m_i \geq 0$, consider the homogeneous ideal $I = \cap_i I(p_i)^{m_i} \subseteq K[\mathbb{P}^N] = K[x_0,\dots,x_N]$. It defines a 0-dimensional subscheme $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$ called a fat point subscheme, where by definition we have $I(Z) = I$. We denote the K-vector space span of the forms of degree t in $I(Z)$ by $[I(Z)]_t$. Let $E_Z = m_1E_1 + \cdots + m_sE_s$.

Exercise 1.2.6. (See [11, Proposition IV.1.1].) Show there is a canonical K-vector space isomorphism

$$H^0(X, \mathcal{O}_X(tL - E_Z)) \cong [I(Z)]_t.$$

Definition 1.2.7. Given a divisor D on a smooth projective surface X, we say D is *semi-effective* if for some $m > 0$ we have $h^0(X, \mathcal{O}_X(mD)) > 0$ (i.e., for some $m > 0$, $|mD| \neq \varnothing$, so mD is linearly equivalent to an effective divisor).

Here is a question raised by Eisenbud and Velasco (2009) regarding semi-effectivity.

Open Problem 1.2.8 (Eisenbud-Velasco)**.** *Given an arbitrary $t \geq 0$ and $E_Z = m_1E_1 + \cdots + m_sE_s$ with $m_i \geq 0$, is there an algorithm to determine whether $tL - E_Z$ is semi-effective (or equivalently $\dim[I(mZ)]_{mt} > 0$ for $Z = m_1p_1 + \cdots + m_sp_s$)?*

1.3. **Waldschmidt constants.**

Eisenbud and Velasco's question can be partially addressed by Waldschmidt constants [19]. Let $I \subseteq K[\mathbb{P}^N]$ be a nonzero homogeneous ideal. We define $\alpha(I)$ to be the least t such that $[I]_t \neq 0$.

Exercise 1.3.1. If $0 \subsetneq I \subseteq J \subseteq K[\mathbb{P}^N]$ are homogeneous ideals, show that $\alpha(IJ) = \alpha(I) + \alpha(J)$. In particular, conclude that $\alpha(I^r) = r\,\alpha(I)$.

As an aside we note that given $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$, its ideal $I = I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s}$, the mth *symbolic power* of I, denoted $I^{(m)}$, is $I^{(m)} = I(Z)^{(m)} = I(mZ) = I(p_1)^{mm_1} \cap \cdots \cap I(p_s)^{mm_s}$. This terminology is often used in the literature. Moreover, one can define symbolic powers of any homogeneous ideal, but doing so involves technicalities, so we will avoid that for now.

Definition 1.3.2. Let $Z = m_1p_1 + \cdots + m_sp_s$ be a nonzero fat point subscheme of $\mathbb{P}^N$. The *Waldschmidt constant* $\widehat{\alpha}(I(Z))$ of $I(Z)$ is

$$\widehat{\alpha}(I(Z)) = \inf\Big\{\frac{\alpha(I(mZ))}{m} : m > 0\Big\}.$$

Exercise 1.3.3. Let Z be a nonzero fat point subscheme of $\mathbb{P}^N$.

(a) Show that $1 \leq \widehat{\alpha}(I(Z)) \leq \sum_i m_i$.

(b) Let m, n be positive integers. Show that

$$\alpha(I((m+n)Z)) \leq \alpha(I(mZ)) + \alpha(I(nZ)).$$

(c) Let m, n be positive integers. Show that

$$\frac{\alpha(I(mnZ))}{mn} \leq \frac{\alpha(I(mZ))}{m}.$$

(d) Use Fekete's Subadditivity Lemma [7] to conclude for each n that

$$\widehat{\alpha}(I(Z)) = \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m} \leq \frac{\alpha(I(nZ))}{n}.$$

(e) Show that $\widehat{\alpha}(I(nZ)) = n\,\widehat{\alpha}(I(Z))$.

(f) Over the complexes, Waldschmidt and Skoda [19, 18] obtained the bound

$$\frac{\alpha(I(Z))}{N} \leq \widehat{\alpha}(I(Z))$$

using some rather hard analysis. A proof using multiplier ideals is given in [15]. Here is another approach (see [12]). It is known that $I((N + m - 1)rZ) \subseteq I(mZ)^r$ for all $m, r > 0$ [5, 14]. Assuming this, show for each $n > 0$ that

$$\frac{\alpha(I(mZ))}{N+m-1} \leq \widehat{\alpha}(I(Z))$$

and hence that

$$\frac{\alpha(I(mZ))}{N+m-1} \leq \widehat{\alpha}(I(Z)) \leq \frac{\alpha(I(mZ))}{m}.$$

Exercise 1.3.4. Let Z be a nonzero fat point subscheme of $\mathbb{P}^N$ and $I = I(Z)$. If $\frac{\alpha(I^{(m)})}{m} \leq \frac{\alpha(I^{(n)})}{n}$, show that

$$\frac{\alpha(I^{(m+n)})}{m+n} \leq \frac{\alpha(I^{(n)})}{n}.$$

Exercise 1.3.5. Let $Z = m_1p_1 + \cdots + m_sp_s$ and $Z' = m'_1p_1 + \cdots + m'_sp_s$ be fat point subschemes of $\mathbb{P}^N$ for distinct points p_i with $0 \leq m_i \leq m'_i$ for all i. Show that $\widehat{\alpha}(I(Z)) \leq \widehat{\alpha}(I(Z'))$. Give an example where $Z \neq Z'$ but $\widehat{\alpha}(I(Z)) = \widehat{\alpha}(I(Z'))$.

Exercise 1.3.6. Let $Z = m_1p_1 + \cdots + m_sp_s$ be a nonzero fat point subscheme of $\mathbb{P}^N$. Show that $\widehat{\alpha}(I(Z)) \leq \sqrt[N]{\sum_i m_i^N}$.

By Exercise 1.3.3(f), it is possible to compute $\widehat{\alpha}(I(Z))$ to any desired number of decimal places by just computing $\alpha(I(mZ))$ for large m. Thus for any real number $a \neq \widehat{\alpha}(I(Z))$, it is possible to computationally verify that $a \neq \widehat{\alpha}(I(Z))$. What is not clear is how to computationally verify that $a = \widehat{\alpha}(I(Z))$ when a in fact does equal $\widehat{\alpha}(I(Z))$.

Corollary 1.3.7. *Let $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$ be a nonzero fat point subscheme. If $t > \widehat{\alpha}(I(Z))$, then $\dim[I(mZ)]_{mt} > 0$ for all $m \gg 0$, and if $t < \widehat{\alpha}(I(Z))$, then $\dim[I(mZ)]_{mt} = 0$ for all $m > 0$.*

Proof. Say $t > \widehat{\alpha}(I(Z))$. Then for $m \gg 0$, we have $mt > \alpha(I(mZ))$, so $\dim[I(mZ)]_{mt} > 0$. If $t < \widehat{\alpha}(I(Z))$, then $mt < m\widehat{\alpha}(I(Z)) \leq \alpha(I(mZ))$ for all m, so $\dim[I(mZ)]_{mt} = 0$. □

In addition to computing Waldschmidt constants, recent work [3] raises the question of how large the least m can be in Problem 1.2.8, given that $h^0(X, \mathcal{O}_X(tmL - mE_Z)) > 0$ for some $m > 0$.

Exercise 1.3.8. Let $r > 1$. Given distinct lines $L_1, \ldots, L_{2r} \subset \mathbb{P}^2$ with $t_k = 0$ for $k > 2$, let $Z = p_1 + \cdots + p_s$ be the t_2 points of intersections of the lines (so $t_2 = \binom{2r}{2}$). Show that $\dim[I(mZ)]_{mr} = 0$ for all odd $m > 0$ and $\dim[I(mZ)]_{mr} = 1$ for all even $m > 0$. Conclude that $\widehat{\alpha}(I(Z)) = r$ and that the least m such that $h^0(X, \mathcal{O}_X(mrL - mE_Z)) > 0$ is $m = 2$. Moreover, $h^0(X, \mathcal{O}_X(2rL - 2E_Z)) = 1$ and the intersection matrix of the components of the unique divisor in $|2(rL - E_Z)|$ is negative definite.

Exercise 1.3.9. Let Z be the reduced scheme consisting of the 9 crossing points defined in Figure 2. Show that the least $m > 0$ such that $\dim[I(m2Z)]_{5m} > 0$ is $m = 2$.

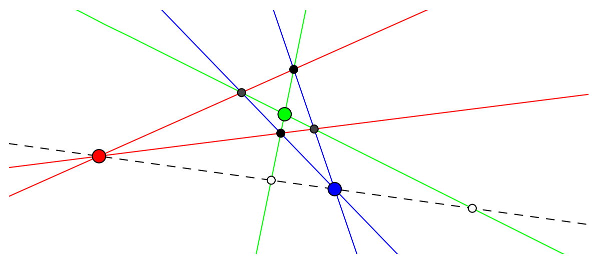


FIGURE 2. Three reducible conics (colored red, green and blue) through 4 general points (the black dots) with their singular points (the three larger colored dots) plus an additional line (dashed black) giving two more points (the open dots) define a set of 9 points.

Example 1.3.10. Consider points p_1, p_2 and p_3 on an irreducible conic C', and the three points p_4, p_5 and p_6 of the conic infinitely near to these first three points, as show in Figure 3 (where the infinitely near points are represented by tangent directions). Blow up all 6 points to get a surface X, let C be the proper transform of C', and let E_i be the blow up of point p_i. Thus $E_i = N_i + E_{i+3}$ for $i = 1, 2, 3$ has two components, as shown. Let L be the pullback of a line from $\mathbb{P}^2$ to X. Let $F = L - E_3 - E_4 - E_6$. Since $F \cdot N_i < 0$, if F were linearly equivalent to an effective divisor, then $F - N_1 - N_2 - N_3 = L - E_1 - E_2 - E_3$ would be also, but it is not, since the points p_1, p_2, p_3 are not collinear.

However, $2F \sim D = C + N_1 + N_2 + N_3$ is linearly equivalent to an effective divisor, and the intersection matrix of the components of D is clearly negative definite.

Exercise 1.3.11. Generalize Example 1.3.10 by replacing the conic with a reduced irreducible curve of degree d to obtain a surface X and a divisor $F = L - E_{i_1} - \cdots - E_{i_t}$ where $t = \binom{d+1}{2}$ such that the least m with $h^0(X, \mathcal{O}_X(mF)) > 0$ is d and the intersection matrix of the effective divisor $D \sim dF$ is negative definite. (This contrasts with other exercises, where either the least m is bounded, or the intersection matrix is not negative definite.)

Exercise 1.3.12. Let $Z = p_1 + \cdots + p_7$ for the 7 points p_i of the Fermat arrangement for $n = 2$. Recall that the Fermat arrangement consists of $n^2 + 3$ points, three of which are the coordinate vertices of $\mathbb{P}^2$; assume that these three are p_5, p_6 and p_7.

(a) Show that $h^0(X, \mathcal{O}_X(3F)) > 0$ for $F = 5L - 2E_Z$; conclude that $\alpha(I(6mZ)) \leq 15m$. (Hint: Show that $|3F|$ contains a curve which is a sum of proper transforms of lines.)
(b) Show that the least $m > 0$ such that $h^0(X, \mathcal{O}_X(mF)) > 0$ is $m = 2$.

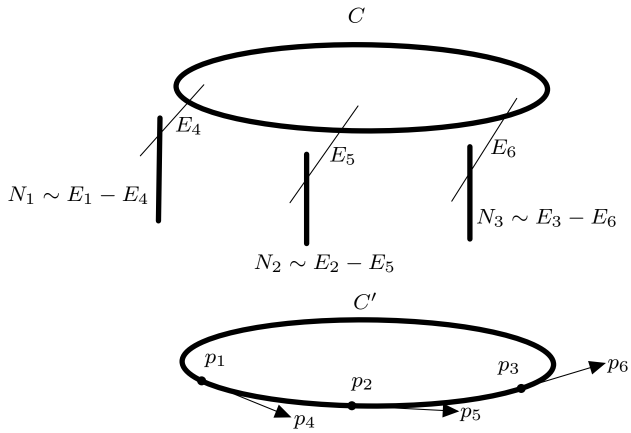


FIGURE 3. A conic with 3 points and 3 infinitely near points blown up.

(c) Show that $H = 4L - E_1 - \cdots - E_4 - 2(E_5 - E_6 - E_7)$ is nef. (Recall that *nef* means $H \cdot C \geq 0$ for every effective divisor C. Hint: Show that $|3H|$ or even $|H|$ contains a curve B' which is a sum of the proper transforms B_i of lines, and that $H \cdot B_i \geq 0$ for each summand.)
(d) Conclude that $\alpha(I(6mZ)) \geq 15m$. (Hint: Note that $H \cdot F = 0$.)
(e) Conclude that $\widehat{\alpha}(I(Z)) = \frac{15}{6} = 2.5$.

Exercise 1.3.13. Let $Z = p_1 + \cdots + p_s$ be the $s = n^2 + 3$ points p_i of the Fermat arrangement for $n > 2$, where $p_{n^2+1} = p_{s-2}, p_{n^2+2} = p_{s-1}, p_{n^2+3} = p_s$ are the coordinate vertices. Let $Y = p_1 + \cdots + p_{n^2}$.
(a) Show that $n = \widehat{\alpha}(I(Y)) \leq \widehat{\alpha}(I(Z))$.
(b) Show that the least $m > 0$ such that $\dim[I(mZ)]_{mn} > 0$ is $m = 3$. Conclude that $\alpha(I(3Z)) \leq 3n$ and thus that $\widehat{\alpha}(I(Z)) = n$.

Exercise 1.3.14. Let $Z = p_1 + \cdots + p_{49}$ be the points of the Klein arrangement. Show that the least $m > 0$ such that $\dim[I(mZ)]_{m7} > 0$ is $m = 3$. Conclude that $\widehat{\alpha}(I(Z)) \leq 7$ and $\widehat{\alpha}(I(3Z)) \leq 21$. (Alternatively, let $X = p_1 + \cdots + p_{21}$ and $Y = p_{22} + \cdots + p_{49}$, where X consists of the $t_4 = 21$ points of the Klein arrangement of multiplicity 4 and Y consists of the $t_3 = 28$ points of the Klein arrangement of multiplicity 3. Let $V = 4X + 3Y$. Show that $\widehat{\alpha}(I(V)) = 21$, and hence by Exercise 1.3.5 that $\widehat{\alpha}(I(3Z)) \leq 21$, so $\widehat{\alpha}(I(Z)) \leq 7$ by Exercise 1.3.3(e).)

Exercise 1.3.15. Let $Z = p_1 + \cdots + p_{201}$ be the points of the Wiman arrangement. Show that the least $m > 0$ such that $\dim[I(mZ)]_{m15} > 0$ is $m = 3$. Conclude that $\widehat{\alpha}(I(Z)) \leq 15$ and $\widehat{\alpha}(I(3Z)) \leq 45$. (Alternatively, let $W = p_1 + \cdots + p_{36}$, $X = p_{37} + \cdots + p_{81}$ and $Y = p_{82} + \cdots + p_{201}$ where W consists of the $t_5 = 36$ points of the Wiman arrangement of multiplicity 5, X consists of the $t_4 = 45$ points of the Wiman arrangement of multiplicity 4 and Y consists of the $t_4 = 45$ points of the Wiman arrangement of multiplicity 3. Let $V = 5W + 4X + 3Y$. Show that $\widehat{\alpha}(I(V)) = 45$, and hence by Exercise 1.3.5 that $\widehat{\alpha}(I(3Z)) \leq 45$, so $\widehat{\alpha}(I(Z)) \leq 15$ by Exercise 1.3.3(e).)

If Z is the reduced scheme of singular points of the Wiman arrangement of 45 lines, then $\widehat{\alpha}(I(Z)) = 27/2$ [2]. The Klein is a little harder, but it is looking like $\widehat{\alpha}(I(Z)) = 13/2$ for the Klein [2].

Open Problem 1.3.16. *Compute $\widehat{\alpha}(I(Z))$ if $Z = \sum_i p_i$ is the reduced scheme consisting of the crossing points of the Klein arrangement of lines.*

The least m can be bigger than just 3, even without using infinitely near points.

Exercise 1.3.17. Assume $\mathrm{char}(K) > 0$. Let $F_q \subset K$ be a subfield of order q. Let $Z = p_1 + \cdots + p_s$ be all but one of the points of $\mathbb{P}^2$ defined over F_q (so $s = q^2 + q$). Show that the least $m > 0$ such that $\dim[I(mZ)]_{mq} > 0$ is $m = q$. Show that $\widehat{\alpha}(I(Z)) = q$.

For additional examples, it is helpful to know the dimension of $[I(Z)]_t$ in each t. For general points in $\mathbb{P}^2$, there is a conjecture for this, the SHGH Conjecture. But first, we put it in context by recalling a general result.

Theorem 1.3.18. *Given a fat point subscheme $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$, we have*

$$\dim[I(Z)]_t \geq \max\Big\{0, \binom{t+N}{N} - \sum_i \binom{m_i+N-1}{N}\Big\},$$

with equality for $t \geq \sum_i m_i - 1$.

Proof. Let $I = I(Z)$. The forms in $[I]_t$ are the solutions to $\sum_i \binom{m_i+N-1}{N}$ homogeneous linear equations (possibly not independent) on the $\binom{t+N}{N}$ dimensional vector space of forms of degree t (i.e., vanishing on Z imposes $\sum_i \binom{m_i+N-1}{N}$ conditions on all forms of degree t), so we get the lower bound on the dimension as claimed.

The equality can be thought of as a form of the Chinese Remainder Theorem. Let $R = K[\mathbb{P}^N]$. Let $S = K[y_1, \ldots, y_N]$, where we think of y_i as x_i/x_0, assuming that the coordinates x_i have been chosen such that x_0 does not vanish at any of the points p_i. If $p_i = (a_0, \ldots, a_n)$, let $q_i = (a_1/a_0, \ldots, a_N/a_0)$. Define $J = J(q_1)^{m_1} \cdots J(q_s)^{m_s}$, where $J(q_i)$ is the ideal of all polynomials in S that vanish at q_i. We have a vector space isomorphism $(S/J)_t \cong [R/I]_t = [R]_t/[I]_t$ given for any polynomial $(y_1, \ldots, y_N)$ of degree at most t by $f(y_1, \ldots, y_N) \mapsto x_0^t f(x_1/x_0, \ldots, x_N/x_0)$, where by $(S/J)_t$ we mean the vector space image under $S \to S/J$ of all polynomials of degree t or less in S.

The ideals $J(q_i)^{m_i}$ are pairwise coprime, so $J = \cap_i J(q_i)^{m_i}$ and $S \cong \oplus_i S/J(q_i)^{m_i}$. Since up to a linear change of coordinates $S/J(q_i)^{m_i}$ is $S/(y_1, \ldots, y_N)^{m_i}$, we see that $\dim S/J(q_i)^{m_i} = \binom{m_i+N-1}{N}$, hence $\dim S/J = \sum_i S/J(q_i)^{m_i} = \sum_i \binom{m_i+N-1}{N}$, so for $t \gg 0$ we have $S/J = (S/J)_t \cong [R/I]_t = [R]_t/[I]_t$, hence $\dim[I(Z)]_t = \dim[R]_t - \dim[R/I]_t = \binom{t+N}{N} - \sum_i \binom{m_i+N-1}{N}$ for $t \gg 0$.

The inverse isomorphism $\sum_i S/J(q_i)^{m_i} \to S/J$ is given by $(f, \ldots, f_s) \mapsto \sum_i f_ig_i$, where we can represent f_i by a polynomial of degree $m_i - 1$ and g_i is represented by a polynomial that doesn't vanish at q_i and is in $\Pi_{j\neq i} J(q_j)^{m_j}$. By picking linear forms L_i that vanish at q_i but not at any other q_j, we can take g_i to be $L_1^{m_1} \cdots L_s^{m_s}/L_i^{m_i}$. Thus $\deg(f_ig_i) = \sum_i m_i - 1$, so for $t = \sum_i m_i - 1$ we have isomorphisms $S/J = (S/J)_t \cong [R/I]_t = [R]_t/[I]_t$, hence $\dim[I(Z)]_t = \binom{t+N}{N} - \sum_i \binom{m_i+N-1}{N}$ for $t \geq \sum_i m_i - 1$. □

When $N = 2$, this also follows from Riemann-Roch for a blow up X of $\mathbb{P}^2$.

Exercise 1.3.19. Given distinct points $p_1, \ldots, p_s \in \mathbb{P}^2$ and integers $t, m_1, \ldots, m_s \geq 0$, let $Z = m_1p_1 + \cdots + m_sp_s$. Using Riemann-Roch and Serre duality with $F = tL - E_Z$, show that

$$h^0(X, \mathcal{O}_X(F)) \geq \frac{F^2 - K_XF}{2} + 1.$$

Conclude that

$$\dim[I(Z)]_t = h^0(X, \mathcal{O}_X(F)) \geq \max\Big\{0, \binom{t+2}{2} - \sum_i \binom{m_i+1}{2}\Big\}.$$

To obtain examples that are not exclusively positive characteristic examples, we now recall a special case of the SHGH Conjecture (see [17, 10, 9, 13] for various equivalent versions of the full conjecture).

Conjecture 1.3.20. *Let $Z = p_1 + \cdots + p_s \subset \mathbb{P}^2$ for general points p_i, where either s is a square or $s > 8$. Then* $\dim[I(mZ)]_t = \max\Big\{0, \binom{t+2}{2} - s\binom{m+1}{2}\Big\}$.

Conjecture 1.3.20 is known to be true when s is a square [6, 4, 16].

Exercise 1.3.21. Consider $Z = p_1 + \cdots + p_{s^2}$ for s^2 general points p_i for $s > 6$. Let $F = (s+1)L - E_Z$. Show that the least m such that $h^0(X, \mathcal{O}_X(mF)) > 0$ is $m = \lceil \frac{s-3}{2} \rceil$.

Conjecturally, similar examples arise where the least m can be arbitrarily large even when the number of points is fixed.

Exercise 1.3.22. (See [3].) Let $s > 49$ not be a square and consider positive integers t and r such that $t^2 - sr^2 = 1$. Let $Z = rp_1 + \cdots + rp_s$ for general points $p_i \in \mathbb{P}^2$. Let $F = tL - E_Z$. Since $F^2 > 0$ and $F \cdot L > 0$, we know $h^0(X, \mathcal{O}_X(mF)) > 0$ for $m \gg 0$. Assuming the SHGH Conjecture, show that the least such m satisfies $m > r(s - 3\sqrt{2s})/2$. (Since there are examples of t and r with $t^2 - sr^2 = 1$ and r arbitrarily large, there is no bound on the least m such that $h^0(X, \mathcal{O}_X(mF)) > 0$.)

1.4. **Zariski Decompositions.** Zariski decompositions were first proved for effective divisors [20] on any smooth projective surface X. See [1] for a simplified proof. A more general version can be found in [8]. Here we prove them for any effective divisor D on a blow up X of the plane. It is not hard to see that it actually is enough to assume D is semi-effective (i.e., tD is linearly equivalent to an effective divisor for some $t > 0$).

Theorem 1.4.1. *Let X be the blow up of a finite set of points of the plane. If $D = m_1N_1 + \cdots + m_rN_r$ where the m_i are positive integers and each N_i is a reduced irreducible curve on X, then we can write $D = P + N$ where $P = a_1N_1 + \cdots + a_rN_r$ is nef, the a_i are nonnegative, $P \cdot N = 0$ and either $N = 0$ or $N = b_{i_1}N_{i_1} + \cdots + b_{i_j}N_{i_s}$ with the b_{i_j} positive and the matrix $(N_{i_j} \cdot N_{i_k})$ negative definite. Moreover, if D' is effective with Zariski decomposition $P' + N'$, and linearly equivalent to D, then P' and P are linearly equivalent and $N' = N$.*

Exercise 1.4.2. Let X be the blow up of r points of the plane. If $N_1, \ldots, N_s$ are prime divisors such that the matrix $(N_i \cdot N_j)$ is negative definite, prove that the N_i are linearly independent in the divisor class group. Conclude that $s \leq r$.

Exercise 1.4.3. Let X be the blow up of the $r = \binom{6}{2} = 15$ points of intersection of 6 general lines in the plane. Let D be the sum of the proper transforms of the 6 lines (so up to linear equivalence $D \sim 6L - 2E_1 - \cdots - 2E_{15}$). Let L be the proper transform of a general line. Find a Zariski decomposition for each of the following divisors: $D_3 = D/2 \sim 3L - E_1 - \cdots - E_{15}$, $D_4 = L + D/2 \sim 4L - E_1 - \cdots - E_{15}$, $D_5 = 2L + D/2 \sim 5L - E_1 - \cdots - E_{15}$, $D_6 = D \sim 6L - 2E_1 - \cdots - 2E_{15}$, and $D_7 = 4L + D \sim 10L - 2E_1 - \cdots - 2E_{15}$.

For the proof of Theorem 1.4.1 we will use a lemma and some exercises.

Exercise 1.4.4. Let $N_1, \ldots, N_r$ be distinct reduced irreducible curves with $N_i^2 < 0$ for all i such that no nonzero nonnegative sum $m_1N_1 + \cdots + m_rN_r$ is nef. Then there is an orthogonal basis $N_1^*, \ldots, N_r^*$ where $N_1^* = N_1$, $N_2^* = c_{21}N_1^* + N_2$, $N_3^* = c_{31}N_1^* + c_{32}N_2^* + N_3$, …, $N_r^* = c_{r1}N_1^* + c_{r2}N_2^* + \cdots + c_{r,r-1}N_{r-1}^* + N_r$ with each c_{ij} rational and $c_{ij} \geq 0$ (so each N_i^* is a nonnegative rational linear combination of the N_j) and $(N_i^*)^2 < 0$ for each i.

Lemma 1.4.5. *Let $N_1, \ldots, N_r$ be reduced irreducible curves. Then the matrix $(N_i \cdot N_j)$ is negative definite if and only if no nonzero nonnegative sum $m_1N_1 + \cdots + m_rN_r$ is nef.*

Proof. Assume the matrix $(N_i \cdot N_j)$ is negative definite. Thus for any nonzero nonnegative linear combination $N = m_1N_1 + \cdots + m_rN_r$ we have $N^2 < 0$ and hence N is not nef.

Conversely, assume no nonzero nonnegative sum $m_1N_1 + \cdots + m_rN_r$ is nef. Thus $N_i^2 < 0$ for all i. Now apply Exercise 1.4.4. Thus the span of $N_1, \ldots, N_r$ has an orthogonal basis where each basis element has negative self-intersection, hence $(N_i \cdot N_j)$ is negative definite. □

Exercise 1.4.6. Let V be a finite dimensional vector space with a positive definite inner product. Let $v_1, \dots, v_r$ be a basis such that $v_i v_j \leq 0$ for all $i \neq j$. If $v \in V$ has $vv_i \geq 0$ for all i, show that $v = a_1 v_1 + \cdots + a_r v_r$ where $a_i \geq 0$ for all i.

Corollary 1.4.7. *Let $N_1, \dots, N_r$ be reduced irreducible curves with $N_i^2 < 0$ for all i such that no nonzero nonnegative sum $m_1 N_1 + \cdots + m_r N_r$ is nef. Then there is a dual basis $N'_1, \dots, N'_r$ where: $N'_i N_j = 0$ for all $i \neq j$; $N'_i N_i = (N'_i)^2 < 0$ for all i; and each N'_i is a nonnegative rational linear combination of the N_j.*

Proof. By Lemma 1.4.5, the intersection form on the span of $N_1, \dots, N_r$ is negative definite. The dual basis elements N'_i are solutions to the linear equations $N'_i N_j = 0$, which are defined over the integers, so the solutions are rational linear combination of the N_j. Negative definiteness gives $(N'_i)^2 < 0$, and $N'_i N_i = (N'_i)^2$ comes down to a choice of scaling. The fact that each N'_i is a *nonnegative* rational linear combination of the N_j comes from Exercise 1.4.6 (after converting the result to the negative definite case). □

Proof of Theorem 1.4.1. Start with $D = M + N$, where $M = 0$ and $N = m_1 N_1 + \cdots + m_r N_r$. If some nonzero nonnegative sum $S = n_1 N_1 + \cdots + n_r N_r$ is nef, let c be the minimum of the ratios m_i/n_i for which $n_i > 0$. Replace M by $M + cS$ and replace N by $N - cS$. Then $D = M + N$ and M and N are still nonnegative sums of the N_i, with M still nef but N having one fewer summand. Repeat this process until either $N = 0$ or N is a sum $N = b_{i_1} N_{i_1} + \cdots + b_{i_j} N_{i_s}$ such that $b_{i_j} > 0$ for all j but no nonnegative sum of the N_{i_j} is nef.

Thus we have $D = M + N$ where M is a nef nonnegative rational sum of the curves N_i, and N is either 0 (and we are done) or a positive rational sum $N = b_{i_1} N_{i_1} + \cdots + b_{i_j} N_{i_s}$ where no nonzero nonnegative sum of the N_{i_j} is nef.

In the latter case, if $MN_{i_j} = 0$ for all i we take $P = M$ and N as is, and we are done. So suppose $MN_{i_j} > 0$ for some i. Consider the dual basis $\{N'_{j_k}\}$ given in Corollary 1.4.7. We can write $N'_{i_j} = \sum_j a_{i_j} N_{i_j}$ with nonnegative a_{i_j}. Choose the maximum t such that $ta_{i_j} \leq b_{i_j}$ for all j and such that $(M + tN'_{i_j})N_{i_j} \geq 0$, and replace M by $M + tN'_{i_j}$ and N by $N - tN'_{i_j}$. Then either the number of basis elements N_{j_k} meeting M positively has gone down by 1 of the number of terms in N has gone down by 1. Repeating this process eventually gives a $P = M$ orthogonal to all terms (if any) of N.

Moreover, if D' is effective with Zariski decomposition $P' + N'$, and linearly equivalent to D, then P' and P are linearly equivalent and $N' = N$.

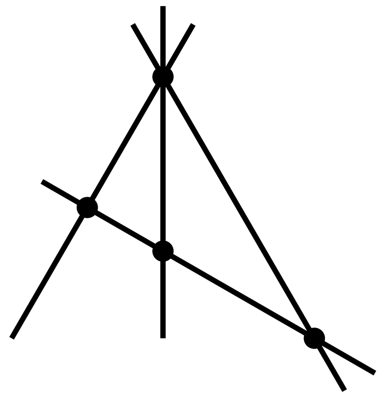
For the uniqueness assertion, pick an integer $t > 0$ such that tP, tP', tN and tN' are all integral. Then some component C_1 of tN has $C_1 \cdot tN < 0$, so $C_1 \cdot tN' \leq C_1 \cdot tD' = C_1 \cdot tD = C_1 \cdot tN < 0$. Thus C_1 is a component of tN'. If $tN \neq C_1$, then for some component C_2 of $tN - C_1$, by negative definiteness we have $C_2 \cdot (tN' - C_1) \leq C_1 2 \cdot (tN - C_1) < 0$. Repeating this, we eventually see that $tN' - tN$ is effective. Reversing the argument shows that $tN' - tN$ is also effective, so $tN' = tN$. Thus $tP' = tD' - tN'$ is linearly equivalent to $tP = tD - tN$, as claimed. □

Computing $\widehat{\alpha}(I)$ can sometimes come from computing Zariski decompositions.

Proposition 1.4.8. *Let $p_1, \cdots, p_r$ be distinct points in the plane and let I be the radical ideal of the points. Let X be the surface obtained by blowing up of the points and let $F = dL - m_1 E_1 - \cdots - m_r E_r$. If F has a Zariski decomposition of the form $P + N$ where $P \neq 0$ and $N = aL - b(E_1 + \cdots + E_r) \neq 0$, then $\widehat{\alpha}(I) = \frac{a}{b}$.*

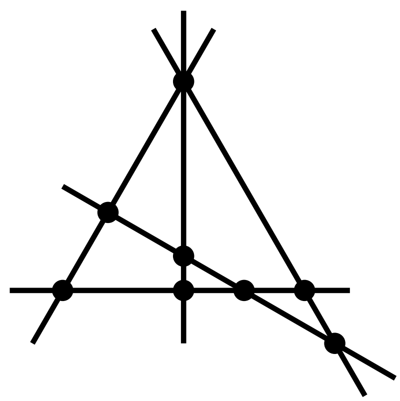
Proof. Since N is effective we have $\widehat{\alpha}(I) \leq \frac{a}{b}$. Let $E = E_1 + \cdots + E_r$. Since P is nef, we have $(PL)b(\widehat{\alpha}(I)) - bPE = P(b(\widehat{\alpha}(I))L - bE) \geq 0 = PN = (PL)a - bPE$, so $b(\widehat{\alpha}(I)) \geq a$ or $\widehat{\alpha}(I)) \geq a/b$. □

Example 1.4.9. Compute $\widehat{\alpha}(I)$ for the ideal I of the points of intersection of the lines in the following figure.



Answer: Let p_1 be the triple point, p_2, p_3, p_4 the other three points. Take $E = E_2 + E_3 + E_4$ and $F = 8L - 5E_1 - 4E$. Its Zariski decomposition is $P = 3L - 2E_1 - E$ and $N = 5L - 3E_1 - 3E$, where N is the sum of the proper transforms of the lines through p_1 and twice the proper transform of the line through the other 3 points. So $\widehat{\alpha}(I) = 5/3$.

Exercise 1.4.10. Use a Zariski decomposition to compute $\widehat{\alpha}(I)$ for the ideal I of the points of intersection of the lines in the following figure.



Exercise 1.4.11. Use a Zariski decomposition to compute $\widehat{\alpha}(I)$ for the ideal I of the points of intersection of $d > 2$ general lines in the plane.

Exercise 1.4.12. Let X be the surface obtained by blowing up points $p_1, \dots, p_r$. Let I be the ideal of the points and let $F_{t,m} = tL - m(E_1 + \cdots + E_r)$. Show that

$$\widehat{\alpha}(I) = \inf\Big\{\frac{t}{m} : h^0(X, \mathcal{O}_X(F_{t,m})) > 0\Big\}.$$

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2. Bounded Negativity Conjecture (BNC) and H-constants

2.1. **Bounded Negativity.** Let X be a smooth projective surface. If C is a curve on X, how negative can C^2 be? This certainly depends on X. For example, for $X = \mathbb{P}^2$ we have $C^2 > 0$ for all C.

Exercise 2.1.1. Let $X \to \mathbb{P}^2$ be the blow up of $n \geq 2$ distinct points $p_1, \dots, p_n$ on a line $L \subset \mathbb{P}^2$. Let L be the pullback of a line and E_i the blow up of p_i. Consider the divisor $F = dL - m_1E_1 - \cdots - m_nE_n$ on X.

(a) Show that $|F|$ is nonempty if and only if $d \geq \max(m_1, \dots, m_n, 0)$.
(b) If D is a divisor on X such that $D \cdot E_i \geq 0$ for all i and $D \cdot H \geq 0$ where H is the proper transform of L (so $H \sim L - E_1 - \cdots - E_n$), show that $D^2 \geq 0$. Conclude that the only reduced irreducible curves C on X with $C^2 < 0$ are $E_1, \dots, E_n$ and H.
(c) Let C be an effective divisor and let m be the multiplicity of the irreducible component of C of maximum multiplicity. Show that $C^2 \geq -m^2n$ and that curves C exist such that equality holds. (Hint: Write $C = P + N$, where P is the sum of the components of C with nonnegative self-intersection, and N is the sum of the irreducible components of C of negative self-intersection, hence $N = a_0H + a_1E_1 + \cdots + a_nE_n$ for some $a_i \geq 0$, so $N^2 \geq -\sum_i(a_i - a_0)^2$. Conclude that $C^2 \geq N^2 \geq -m^2n$.)

This brings us to the Bounded Negativity Conjecture, an old still open folklore conjecture that goes back at least to F. Enriques. There are various versions of the BNC. Here's one.

Conjecture 2.1.2. *Let X be a smooth projective surface, either rational or complex. Then there is a bound B_X such that for any effective divisor C on X, we have $C^2/m^2 \geq B_X$, as long as m is a positive integer at least as big the multiplicity of every component of C.*

Here's another.

Conjecture 2.1.3. *Let X be a smooth projective surface, either rational or complex. Then there is a bound B_X such that for any effective reduced divisor C on X, we have $C^2 \geq B_X$.*

And one more:

Conjecture 2.1.4. *Let X be a smooth projective surface, either rational or complex. Then there is a bound b_X such that for any effective reduced irreducible divisor C on X, we have $C^2 \geq b_X$.*

Over field of positive characteristic bounded negativity can fail; see [6, Exercise V.1.10]. But no counterexamples are known for rational surfaces in any characteristic or for smooth complex projective surfaces.

All three versions of the BNC given above are equivalent. For the equivalence of the second and third, see [3, Proposition 5.1], which given the bound b_X in fact shows that $B_X \leq (\rho(X) - 1)b_X$ suffices, where $\rho(X)$ is the Picard number of X.

Now we show that the first and second versions are equivalent.

Theorem 2.1.5. *Conjecture 2.1.2 holds for X if and only if Conjecture 2.1.3 holds for X, using the same bound B_X.*

Proof. Clearly, Conjecture 2.1.2 implies Conjecture 2.1.3. Conversely, given an effective divisor $C = m_1C_1 + \cdots + m_nC_n$, we have $C^2/m^2 \geq (C_{i_1} + \cdots + C_{i_r})^2 \geq B_X$ for some subset of components C_{i_j}, by Lemma 2.1.6, where m is the maximum of the m_i. □

Lemma 2.1.6. *Let X be a smooth projective surface. Let $C = m_1C_1 + \cdots + m_nC_n$ for distinct reduced irreducible curves C_i on X and integers $m \geq m_i > 0$ with $m = \max(m_1, \dots, m_n)$. Then for some nonempty subset $C_{i_1}, \dots, C_{i_r}$ of the components C_i we have $C^2 \geq m^2(C_{i_1} + \cdots + C_{i_r})^2$.*

Proof. If $C \cdot C_i \geq 0$ for all i, we may assume that $m = m_n$, and then $C^2 \geq m^2 C_n^2$, so assume that $C \cdot C_i < 0$ for some i. Let $P = \sum_{C \cdot C_i \geq 0} m_i C_i$ and $N = \sum_{C \cdot C_j < 0} m_j C_j$. Note that $PN \geq 0$ since P and N have no components in common. Then $C^2 = CP + CN \geq CN = PN + N^2 \geq N^2$. Note that $N \cdot C_j < 0$ for each C_j that appears in N.

It now is enough to prove the claim for N, so we are reduced to the case that $C = m_1 C_1 + \cdots + m_n C_n$ with $C \cdot C_i < 0$ for all i and $m \geq m_i$ for all i. We have $0 > C \cdot m_i C_i \geq C \cdot m C_i$, hence $0 > C^2 = C \cdot \sum_i m_i C_i \geq C \cdot \sum_i m C_i = mC \cdot \sum_i C_i$. Now write $C = P + N$ where now P is the sum of the terms $m_j C_j$ in C such that $C_j \cdot \sum_i C_i \geq 0$ and N is the sum of those terms $m_j C_j$ with $C_j \cdot \sum_i C_i < 0$. Let Q be the same as N except where the coefficient m_j of C_j in each term is replaced by 1. Then $0 > C \cdot \sum_i C_i = (P + N) \cdot \sum_i C_i \geq N \cdot \sum_i C_i \geq mQ \cdot \sum_i C_i \geq mQ^2$. Thus $C^2 \geq C \cdot m \sum_i C_i \geq m^2 Q^2$. □

Example 2.1.7. Given an effective divisor $C = m_1 C_1 + \cdots + m_n C_n$ it's clear in general that $C^2 \geq m^2 (C_1 + \cdots + C_n)^2$ is false, when m is the maximum of the m_i. Take $C = L_1 + 2L_2$ for two different lines L_i in the plane. Then $C^2 = 9$, but $2^2 (L_1 + L_2)^2 = 16$. However, in the proof of Lemma 2.1.6, we reduce to the case that $C = m_1 C_1 + \cdots + m_n C_n$ with $C \cdot C_i < 0$ for all i. One might hope in this case that $C^2 \geq m^2 (C_1 + \cdots + C_n)^2$, but alas no. Blow up the 11 points shown in Figure 4 and let A and B be the proper transforms of A' and B'. Then $(A + 2B)^2 = A^2 + 4AB + 4B^2 = -6 < -4 = 2^2 (A + B)^2$. However we do have $(A + 2B)^2 \geq 2^2 B^2$.

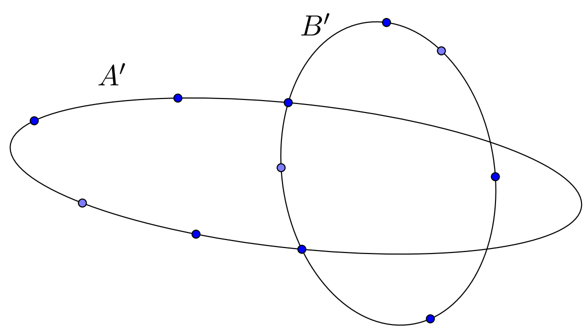

FIGURE 4. Two conics, A and B, in the plane, one through 6 points, the other through 7 points, giving 11 points with 2 in common.

Zariski decompositions provide an easy way to prove Conjectures 2.1.3 and 2.1.4 are equivalent.

Proof. Certainly, if self-intersections of reduced curves are bounded below, then so are the self-intersections of irreducible curves on X. Conversely, let D be any reduced effective divisor. Write $D = C_1 + \cdots + C_r$ where the C_i are distinct reduced, irreducible curves. Let $D = P + N$ be a Zariski decomposition where $N = n_1 N_1 + \cdots + n_s N_s$ with $0 \leq n_i \leq 1$ rational for all i and the N_i prime divisors of negative self-intersection. Then $D^2 \geq N^2 \geq (n_1 N_1)^2 + \cdots + (n_s N_s)^2 \geq N_1^2 + \cdots + N_s^2 \geq (\rho(X) - 1) \min_i \{N_i^2\}$, where $\rho(X) = |S| + 1$ is the Picard number of X (i.e., the rank of the divisor class group). The last inequality comes from Exercise 1.4.2. □

2.2. ***H*-constants.** Given the longstanding difficulty of resolving BNC, it is worth considering variations on the problem, such as the problem of H-constants. A number of different versions have been defined [2, 5, 8, 11]. Here we define them for any curve (typically they have been defined for reduced curves).

Definition 2.2.1. Let $C_1, \ldots, C_r$ be distinct reduced irreducible plane curves and let $C = m_1 C_1 + \cdots + m_r C_r$ where $m_i > 0$ are integers with $m = \max(m_1, \ldots, m_r)$. Then for any nonempty finite subset $S \subset \mathbb{P}^2_K$ we define

$$H(C,S) = \frac{d^2 - \sum_{p\in S}(\mathrm{mult}_p C)^2}{m^2|S|}.$$

We also define

$$H(C) = \inf\Big\{H(C,S) : S \subset \mathbb{P}^2, 0 < |S| < \infty\Big\},$$

$$H_{red}(\mathbb{P}^2_K) = \inf\Big\{H(D) : D \text{ is a reduced curve in } \mathbb{P}^2_K\Big\},$$

$$H_{rir}(\mathbb{P}^2_K) = \inf\Big\{H(D) : D \text{ is a reduced, irreducible curve in } \mathbb{P}^2_K\Big\}$$

and

$$H(\mathbb{P}^2_K) = \inf\Big\{H(D) : D \text{ is a curve in } \mathbb{P}^2_K\Big\}$$

(Clearly $H(\mathbb{P}^2_K) \leq H_{red}(\mathbb{P}^2_K) \leq H_{rir}(\mathbb{P}^2_K)$.)

Exercise 2.2.2. Let C be a plane curve. Show that

$$\inf\Big\{H(C,S) : S \subset C, 0 < |S| < \infty\Big\} = \inf\Big\{H(C,S) : S \subset \mathbb{P}^2, 0 < |S| < \infty\Big\}.$$

Theorem 2.2.3. *If $H_{rir}(\mathbb{P}^2_K) > -\infty$, then Conjecture 2.1.2 holds for every smooth projective rational surface X over the field K.*

Proof. Consider a birational morphism $Y \to X$ of smooth projective surfaces. Let C' be a reduced, irredcuible curve on X and C its proper transform on Y. Then $C^2 \geq (C')^2$. Thus if Conjecture 2.1.4 holds for Y, it also holds for X. However, if X is rational, it is a blow up of points (possibly infinitely near) on a Hirzebruch surface H_n for some n. By blowing up n general points of H_n, we obtain a surface that is also obtained by blowing up distinct points of $\mathbb{P}^2$. (Note, for example, that by blowing up n points p_i on a line in $\mathbb{P}^2$ and any point p off that line, we get a surface B which by contracting the proper transforms of the lines through p and each p_i gives a birational morphism $B \to H_n$.) Thus by blowing up n general points of X we get a birational morphism $Y \to X$, where there is also a birational morphism $Y \to \mathbb{P}^2$ obtained by blowing up a finite set S of distinct points of $\mathbb{P}^2$.

If Conjecture 2.1.4 did not hold for Y, there would be an infinite sequence $C_1, C_2, \dots$ of reduced, irreducible curves on Y such that $C_1^2 > C_2^2 > \cdots$. In all but finitely many cases, C_i maps to a plane curve D_i under $Y \to \mathbb{P}^2$, and so C_i is the proper transform of D_i, hence we have $C_i^2/|S| = (\deg(D_i)^2 - \sum_{p\in S}(\mathrm{mult}_p D_i)^2)/|S| = H(D_i, S)$, which implies $H_{rir}(\mathbb{P}^2) = -\infty$. Thus $H_{rir}(\mathbb{P}^2_K) > \infty$ implies Conjecture 2.1.4 which in turn implies Conjecture 2.1.2. □

Example 2.2.4. In fact $H_{red}(\mathbb{P}^2_K) = -\infty$ if $\mathrm{char}(K) = p > 0$. Let C be the union of all of the lines in $\mathbb{P}^2$ defined over a finite field $F_q \subset K$ of order q. There are q^2+q+1 such lines with q^2+q+1 crossing points, and each point lies on $q+1$ lines. Let S be the points. Then $H(C,S) = -q$, so $H_{red}(\mathbb{P}^2_K) = -\infty$.

Open Problem 2.2.5. *Is $H(\mathbb{P}^2_K) = H_{red}(\mathbb{P}^2_K)$ true for all K?*

Open Problem 2.2.6. *Is $H_{red}(\mathbb{P}^2_{\mathbb{C}}) = -4$? We know $H_{red}(\mathbb{P}^2_{\mathbb{C}}) \leq -4$ due to sequences C_n of reducible curves whose components are plane cubics (see* [9, 10, 4]*), but no complex plane curve C is known with $H(C) \leq -4$. Thus it is of interest to find some examples or show that none exist.*

Open Problem 2.2.7. *Is $H_{rir}(\mathbb{P}^2_K) = -2$? In fact, there is no irreducible plane curve C known over any K with $H(C) \leq -2$.*

Exercise 2.2.8. Show that $H_{rir}(\mathbb{P}^2_K) \leq -2$ over any K by giving a sequence of irreducible curves C_n with $\lim_{n\to\infty} H(C_n) = -2$.

Exercise 2.2.9. If C is a smooth plane curve, $m \geq 1$ and S any nonempty finite subset of C, show that $H(mC) = -1 < H(mC, S)$.

Exercise 2.2.10. If C is a reduced plane curve, $m \geq 1$ and S any nonempty finite subset of smooth points of C, show that $H(mC) \leq -1 < H(mC, S)$.

Exercise 2.2.11. If C is any plane curve, show that $-\infty < H(C) \leq -1$.
(Hint: Show $\min\{-\max\{m_1^2, \ldots, m_n^2, 0\}, -1\} \leq H(C)$, where the m_i are the multiplicities, if any, of the singular points of the reduced curve $\mathrm{red}(C)$.)

Theorem 2.2.12. *Let C be a reduced singular plane curve of some degree d, let T be the set of singular points of C. Then $H(C) < -1$ if and only if $|T| > 0$ and $H(C,T) < -1$, in which case $H(C) = H(C,U)$ for some nonempty subset $U \subseteq T$.*

Proof. First, assume $|T| > 0$ and $H(C,T) < -1$. Then clearly $H(C) < -1$, since $H(C)$ is an infimum over all finite subsets of C. Conversely, first assume $|T| = 0$. Then C is smooth, so $H(C) = -1$ by Exercise 2.2.9.

Next, assume $|T| > 0$ but $H(C,T) \geq -1$. Let $|T| = t$ and let $m_1, \ldots, m_t$ be the multiplicities of C at these points. Let S be a finite set of smooth points of C; let $s = |S|$. Then $H(C,T) = (d^2 - \sum_i m_i^2)/t \geq -1$, so $H(C, S \cup T) = (d^2 - s - \sum_i m_i^2)/(s+t) \geq (-t-s)/(s+t) = -1$. Also, if $s > 0$, then $H(C,S) > -1$ by Exercise 2.2.10.

Now assume $t > 0$ and $H(C,T) \geq -1$, and let $U \cup V = T$ be a disjoint union of nonempty subsets. Let $u = |U|$, $v = |V|$ and m_p be the multiplicity of C at a point p. Then $H(C,U) = (d^2 - \sum_{p \in U} m_p^2)/u$. If this were less than -1, then $-1 \leq H(C,T) = (d^2 - \sum_{p\in U} m_p^2 - \sum_{p \in V} m_p^2)/(u+v) < (-u - \sum_{p\in V} m_p^2)/(u+v) \leq (-u-4v)/(u+v) < -1$. Thus $H(C,U) \geq -1$ for every nonempty subset $U \subseteq T$. Now arguing as before for finite any set of smooth points S of C we have $H(C, S \cup U) \geq -1$. Thus $H(C) \geq -1$.

Finally, assume $H(C) < -1$. Thus there are finite subsets W of C with $H(C,W) < -1$. For any finite subset S of smooth points we saw $H(C,S) > -1$, so W must include points from T. Write W as a disjoint union $W = S \cup U$ where $U \subseteq T$ and the points in S are smooth. If $H(C,U) \geq -1$, then we saw above that we would have $H(C,W) = H(C, S \cup U) \geq -1$. Thus $H(C,U) < -1$, and so $H(C,W) = H(C, S\cup U) = (d^2 - s - \sum_{p \in U} m_p^2)/(s+u) = (-s + uH(C,U))/(s+u) > H(C,U)$, where the last inequality is because $-1 > H(C,U)$. Thus the least values of H come from subsets of T, but T is finite so the infimum is a minimum, and this minimum is attained for a subset of T. □

Open Problem 2.2.13. *Is there an example of a singular plane curve C such that $H(C,U) < H(C,T)$ for some nonempty proper subset U of the set T of singular points of* $\mathrm{red}(C)$*?*

Exercise 2.2.14. If C is a reduced singular plane curve C such that $H(C,U) < H(C,T)$ for some nonempty proper subset U of the set T of singular points of C, show that $H(C,T) < -4$.

Given Open Problem 2.2.7, attention turned to the opposite extreme, curves which are unions of lines [2, 11]. Here are the main facts (see [2]). Define

$$H_{rlin}(\mathbb{P}^2_K) = \inf \Big\{ H(D) : D \text{ is a reduced union of lines in } \mathbb{P}^2_K \Big\}.$$

We have:

$$-2.6 \geq H_{rlin}(\mathbb{P}^2_{\mathbb{Q}}) \geq -3,$$

$$H_{rlin}(\mathbb{P}^2_{\mathbb{R}}) = -3,$$

and

$$-3.358 > -\frac{225}{67} \geq H_{rlin}(\mathbb{P}^2_{\mathbb{C}}) \geq -4.$$

The bound $-2.6 \geq H_{rlin}(\mathbb{P}^2_{\mathbb{Q}})$ comes from taking horizontal, vertical and diagonal lines. The equality $H_{rlin}(\mathbb{P}^2_{\mathbb{R}}) = -3$ comes from $H_{rlin}(\mathbb{P}^2_{\mathbb{R}}) \geq -3$ (apply Theorem 1.1.2) and by giving examples $H(C)$ approaching -3 (there are lots; e.g., regular polygons with their lines of bilateral symmetry). The bound $-\frac{225}{67} \geq H_{rlin}(\mathbb{P}^2_{\mathbb{C}})$ comes from the Wiman arrangement. The bound $H_{rlin}(\mathbb{P}^2_{\mathbb{C}}) \geq -4$ comes from applying an inequality due to Hirzebruch [7]: given any complex arrangement of $n > 3$ lines line such that $t_n = t_{n-1} = 0$, we have

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(k-4)t_k.$$

Exercise 2.2.15. Let $L_1, \dots, L_d$ be distinct lines in the $\mathbb{P}^2_K$. Assume that neither the lines nor any subset of $d-1$ of the lines are concurrent. Also assume that $t_2 = 0$. Let C be the curve given by the union of the lines. Let S be the set of the singular points of C, and set $s = |S|$.

(a) Show that $H(C, S) \leq -2$. Give an example where equality holds.
(b) If $K = \mathbb{C}$, show that $d \leq 3s/4$.
(c) If $K = \mathbb{C}$, show that $H(C, S) \leq -2.25$. Give an example where equality holds.

Open Problem 2.2.16. *Can more be said about $H_{rlin}(\mathbb{P}^2_{\mathbb{Q}})$ and $H_{rlin}(\mathbb{P}^2_{\mathbb{C}})$?*

2.3. **Another formulation of bounded negativity.** Let X be the blow up of the plane at a finite set of points S. We say that X has bounded Zariski denominators if there is an integer d such that for for each divisor D and integer $t > 0$ such that tD is linearly equivalent to an effective divisor, there is an integer $0 \leq e \leq d$ such that the Zariski decomposition $etD = P + N$ has integral divisors P and N .

We now state a version of the main theorem of [1].

Theorem 2.3.1. *Let X be the blow up of the plane at a finite set of points S. Then bounded negativity holds on X (i.e., the set of self-intersections C^2 of reduced curves on X is bounded below) if and only if X has bounded Zariski denominators.*

An exercise will be helpful.

Exercise 2.3.2. Let X be a blow up of the plane at s points. Let $C = dL - m_1E_1 - \cdots - m_sE_s$ be any divisor with $C^2 < 0$ and let the gcd of $d, m_1, \dots, m_s$ be g. Then there is an ample divisor F such that FC and C^2 have gcd g.

Proof of Theorem 2.3.1. Assume bounded negativity holds on X; i.e., $C^2 \geq -b$ for some b and every irreducible curve C. Let $D = d_1D_1 + \cdots + d_rD_r$ be effective (so each d_i is positive and each D_i is a prime divisor) with Zariski decomposition $D = P + N$. Then P and N are sums of the D_i with nonnegative rational coefficients. Since D is integral, the largest denominator used for P is also the largest denominator used for N, so it's enough to look at N. Say $N = n_1N_1 + \cdots + n_sN_s$ where each n_i is positive rational and each N_i is a prime divisor of negative self-intersection. Note that $DN_i = (n_1N_1 + \cdots + n_sN_s)N_i$ gives linear equations for the n_i. The solution involves dividing by $\det(N_iN_j)$, so the largest possible denominator is $\det(N_iN_j)$, but $|\det(N_iN_j)| \leq |N_1^2 \cdots N_s^2|$ (since the volume of a parallelepiped with edges of fixed length is most when the edges are orthogonal). By Exercise 1.4.2, we have $s \leq |S|$. Thus the largest possible denominator is $|b|^{|S|}$, where b is a lower bound for self-intersections of irreducible curves on X.

Conversely, assume X has bounded Zariski denominators, with bound b. Let $C \sim dL - m_1E_1 - \cdots - m_rE_r$ be any prime divisor with $C^2 < 0$. Let $C = gD$ where g is the gcd of $d, m_1, \dots, m_r$. Now say C is a prime divisor with $C^2 < 0$, and pick D to be primitive (i.e., not linearly equivalent to tD' for any integral divisor D' with t an integer bigger than 1) such that $C = gD$. By Exercise 2.3.2 we can pick an ample divisor F such that $FC = g$. Since the Zariski decomposition of D is $D = C/g$, we have $g \leq b$. But for large m, the Zariski decomposition of $F + mC$ is $P = F + (m-a)C$ and $N = aC$

for some a, so $a = (CF + mC^2)/C^2$, hence the denominator needed here is $C^2/\gcd(CF, C^2) \leq b$, hence $C^2 \leq b\gcd(CF, C^2) = bg \leq b^2$. □

Exercise 2.3.3. Let X be the blow up of the plane at a finite number of points such that there is a finite list $A = \{a_1, \ldots, a_r\}$ such that for every prime divisor D with $D^2 < -1$ we have $D^2 \in A$. Assume that there are at most n_i distinct divisors D with $D^2 = a_i$ for each i with $a_i < -1$. Show that no denominator bigger than $|a_1^{n_1} \cdots a_r^{n_r}|$ is ever needed for a Zariski decomposition on X.

Exercise 2.3.4. Determine the largest denominator needed for a Zariski decomposition when X is the blow up of r colinear points of the plane.

Exercise 2.3.5. Determine the largest denominator needed for a Zariski decomposition when X is the blow up of $r+1$ points of the plane on a line L_1 and $s+1$ points on a different line L_2, where one of the points is the point of intersection of the two lines. Assume $r-1$ and $s-1$ are coprime and each is at least 2. [Hint: By "adjunction", we have $C^2 \geq -2 + C(3L - E_1 - \cdots - Er + s + 1)$; see [1, Example 3.2].]

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3. Containment Problems

3.1. **Powers and symbolic powers.** Given distinct points $p_i \in \mathbb{P}^N$, let $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$ be a fat point subscheme. Recall that the mth symbolic power of $I(Z)$ is $I(Z)^{(m)} = I(p_1)^{m_im} \cap \cdots I(p_s)^{mm_i}$. It is interesting to compare this with the rth ordinary power $I(Z)^r = (I(p_1)^{m_1} \cap \cdots I(p_s)^{m_s})^r$ for various m and r. A useful fact here is that $I(Z)^r = Q \cap I(rZ)$ for some M-primary ideal Q, where $M = (x_0, \dots, x_N)$. In particular, Q contains a power of M, hence $[Q]_t = [M]_t$ for all $t \gg 0$, hence $[I(Z)^r]_t = [I(rZ)]_t$ for all $t \gg 0$.

Exercise 3.1.1. Let $I = I(Z)$ for $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$ with $m_i > 0$ for all i.

(a) Show that $I^r \subseteq I^m$ if and only if $r \geq m$.
(b) Show that $I^m \subseteq I^{(m)}$.
(c) If $r \geq m$, show that $I^r \subseteq I^{(m)}$ and $I^{(r)} \subseteq I^{(m)}$.
(d) If $m > r$, show that $I^r \not\subseteq I^{(m)}$. (Hint: look at dimensions of $[K[\mathbb{P}^N]/I^{(m)}]_t$ and $[K[\mathbb{P}^N]/I^r]_t$ for $t \gg 0$.)
(e) Conclude that $I^r \subseteq I^{(m)}$ if and only if $r \geq m$.
(f) If $m < r$, show that $I^{(m)} \not\subseteq I^r$.

It is a subtle and generally open problem to determine for which m and r we have $I^{(m)} \subseteq I^r$, but for $m \gg 0$ we always do have containment:

Exercise 3.1.2. Let $I = I(Z)$ for $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$. Given $r > 0$, let t_r be the least t such that $\dim[K[\mathbb{P}^N]/I^r]_t = \deg(rZ)$. If $m \geq \max\{r, t_r\}$, show that $I^{(m)} \subseteq I^r$.

Let $I = I(Z)$ for some fat point scheme $Z \subset \mathbb{P}^N$. As a refinement of Exercise 3.1.2, we define the saturation degree of I^r: $\mathrm{satdeg}(I^r)$ is the least t such that $(I^r)_j = (I^{(r)})_j$ for all $j \geq t$.

Exercise 3.1.3. Let $I = I(Z)$ for $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^N$, and let t_r be as defined in Exercise 3.1.2. Show that $t_r \geq \mathrm{satdeg}(I^r)$.

Proposition 3.1.4. *Let $I = I(Z)$ be a fat point scheme $Z \subset \mathbb{P}^N$. If $m \geq \max(\mathrm{satdeg}(I^r), r)$, then $I^{(m)} \subseteq I^r$.*

Proof. Since $m \geq r$, we have $I^{(m)} \subseteq I^{(r)}$. Since $m \geq \mathrm{satdeg}(I^r)$, if $[I^{(m)}]_t \neq 0$, then $t \geq m \geq \mathrm{satdeg}(I^r)$, so $[I^{(m)}]_t \subseteq [I^{(r)}]_t = [I^r]_t$. Hence $I^{(m)} \subseteq I^r$. □

Example 3.1.5. The quantity $\mathrm{satdeg}(I^r)$ in Proposition 3.1.4 can be quite large. Consider the case that Z consists of r^2 general points in the plane. Then it is known that $\dim I(mZ)_t = \max(0, \binom{t+2}{2} - r^2\binom{m+1}{2})$ by [7, 13, 19]. One can now check that $\alpha(I(Z)^r) = r\alpha(I(Z)) > \frac{6}{5}r^2 \geq \alpha(I(rZ)) \geq \widehat{\alpha}(I(Z))$ for $r \geq 3$. But $\alpha(I(Z)^r) > \alpha(I(rZ))$ implies $\mathrm{satdeg}(I(Z)^r) \geq \alpha(I(Z)^r) > \frac{6}{5}r^2$. Thus to apply Proposition 3.1.4 to Z here, m cannot be less than or equal to $\frac{6}{5}r^2$.

A formerly open question was:

Question 3.1.6. Given $I = I(Z)$ for a fat point subscheme $Z \subset \mathbb{P}^N$, we know for each r there is an n such $m \geq rn$ implies $I^{(m)} \subseteq I^r$ (take $n = \max\{1, t_r/r\}$), but is there one n that works for all r? And is there an n also independent of Z?

Motivated by [20], the papers [12, 17] give a very general answer, given in Theorem 3.1.7. To see why this result is so surprising, compare the bound it gives for containment (namely, $m \geq Nr$) to the bound required for containment by Proposition 3.1.4, which by Example 3.1.5 may require m to be larger than $6r^2/5$.

Theorem 3.1.7. *Let $I \subseteq K[\mathbb{P}^N]$ be a homogeneous ideal I. Then $I^{(r(m+N-1))} \subseteq (I^{(m)})^r$, and if $m \geq rN$, then (with an appropriate definition of symbolic power when I is not a radical ideal of a finite set of points) we have $I^{(m)} \subseteq I^r$.*

The question now became: is this result optimal? There are various approaches to this question. Here's one showing no constant less than N suffices [6]:

Theorem 3.1.8. *If $c < N$, there is an $r > 0$ and $m > cr$ such that $I^{(m)} \not\subseteq I^r$ for some $I = I(Z)$, where $Z = p_1 + \cdots + p_s \subset \mathbb{P}^N$ for distinct points p_i.*

Exercise 3.1.9. Let $Z \subset \mathbb{P}^N$ be a fat point subscheme, $I = I(Z)$. If $\alpha(I^{(m)}) < r\alpha(I)$, show that $I^{(m)} \not\subseteq I^r$.

Exercise 3.1.10. Pick $s > 2$ lines in $\mathbb{P}^2$ with $t_2 = 0$. For simplicity, assume s is even. Let Z be the $\binom{s}{2}$ crossing points and take $I = I(Z)$. If $m < 2r$ (again for simplicity, assume m is even), show that $I^{(m)} \not\subseteq I^r$ for $s \gg 0$. This shows that there is no $c < 2$, such that $m \geq cr$ is enough to guarantee that $I^{(m)} \subseteq I^r$. A similar construction holds for $\mathbb{P}^N$. (Hint: see Exercise 1.3.8.)

3.2. **The resurgence.** Although $m \geq Nr$ is optimal as a universal bound for homogeneous ideals in $K[\mathbb{P}^N]$, what can one say about bounds for a specific ideal? This question leads to the definition of an asymptotic quantity, the resurgence [6].

Definition 3.2.1. Given a fat point scheme $Z \subset \mathbb{P}^N$, define the *resurgence* $\rho(I)$ for $I = I(Z)$ to be

$$\rho(I(Z)) = \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\right\}.$$

The following result is from [6]. For this we need a new quantity, the regularity.

Definition 3.2.2. The *regularity* $\operatorname{reg}(I)$ of $I = I(Z)$ for a fat point subscheme $Z \subset \mathbb{P}^N$. is defined by specifying that $\operatorname{reg}(I) - 1$ is the least t such that $\dim[I]_t = \binom{t+2}{2} - \deg(Z)$.

Fact 3.2.3. *Let I be the ideal of a fat point subscheme of projective space. An important fact about $\operatorname{reg}(I)$ is that $[I^r]_t = [I^{(r)}]_t$ for $t \geq r\operatorname{reg}(I)$ (see [6]). Another is that I has a set of homogeneous generators each of which has degree at most $\operatorname{reg}(I)$ [9].*

Theorem 3.2.4. *Let $I = I(Z)$ for a nonempty fat point subscheme $Z \subset \mathbb{P}^N$.*

(a) *We have $1 \leq \rho(I) \leq N$.*
(b) *If $m/r < \frac{\alpha(I)}{\widehat{\alpha}(I)}$, then for all $t \gg 0$ we have $I^{(mt)} \not\subseteq I^{rt}$.*
(c) *If $m/r \geq \frac{\operatorname{reg}(I)}{\widehat{\alpha}(I)}$, then $I^{(m)} \subseteq I^r$.*
(d) *We have*

$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I) \leq \frac{\operatorname{reg}(I)}{\widehat{\alpha}(I)},$$

hence $\frac{\alpha(I)}{\widehat{\alpha}(I)} = \rho(I)$ if $\alpha(I) = \operatorname{reg}(I)$.

Proof. (a) By Theorem 3.1.7, we have $\rho(I) \leq N$. By Exercise 3.1.1(f), we have $\rho(I) \geq 1$.

(b) If $m/r < \frac{\alpha(I)}{\widehat{\alpha}(I)}$, then $m\widehat{\alpha}(I) < r\alpha(I)$, so for $t \gg 0$ we have $mt\widehat{\alpha}(I) \leq mt\alpha(I^{(mt)}) < rt\alpha(I)$, hence $I^{(mt)} \not\subseteq I^{rt}$ by Exercise 3.1.9.

(c) Now say $m/r \geq \frac{\operatorname{reg}(I)}{\widehat{\alpha}(I)}$. Then $\alpha(I^{(m)}) \geq m\widehat{\alpha}(I) \geq r\operatorname{reg}(I)$. If $t < \alpha(I^{(m)})$, then $[I^{(m)}]_t = (0) \subseteq I^r$. If $t \geq \alpha(I^{(m)})$, then $t \geq r\operatorname{reg}(I)$ hence $[I^{(m)}]_t \subseteq [I^{(r)}]_t = [I^r]_t$. Thus $I^{(m)} \subseteq I^r$.

(d) This follows from (b) and (c). □

No examples are known with $\rho(I) = N$, but there are a lot of examples with $\rho(I) = 1$. For example, if $|Z| = 1$, so Z consists of a single reduced point, then $\rho(I) = 1$, since $I^{(m)} = I^m$, but it is not not known if $\rho(I) = 1$ guarantees that $I^m = I^{(m)}$ for all m.

An asymptotic version of the resurgence was introduced in [14].

Definition 3.2.5. Given a fat point scheme $Z \subset \mathbb{P}^N$, define the *asymptotic resurgence* $\widehat{\rho}(I)$ for $I = I(Z)$ to be

$$\widehat{\rho}(I(Z)) = \sup\Big\{\frac{m}{r} : I^{(ms)} \not\subseteq I^{rs} \text{ for } s \gg 0\Big\}.$$

In contrast to the case of the resurgence, the result of the following exercise holds not just for ideals I of points, which is one advantage of the asymptotic resurgence (see [14]).

Exercise 3.2.6. Let $Z \subset \mathbb{P}^N$ be a fat point subscheme and let $I = I(Z) \subset \mathbb{P}^N$.

(a) Show that $1 \leq \widehat{\rho}(I) \leq \rho(I)$.

(b) Show that
$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \widehat{\rho}(I) \leq \frac{\omega(I)}{\widehat{\alpha}(I)},$$
where $\omega(I)$ is the maximal degree among a minimal set of homogeneous generators of I. (Hint: Mimic the proof of Theorem 3.2.4(d), using the fact that there is a constant c such that $\operatorname{reg}(I^s) \leq s\omega(I) + c$ for all $s > 0$ [18].)

3.3. **Other perspectives on optimality.** By Theorem 3.1.8, the bound $m \geq rN$ in Theorem 3.1.7 is optimal, in the sense that N cannot be replaced by a smaller number and always still have the containment $I^{(m)} \subseteq I^r$. But given the containment $I^{(Nr)} \subseteq I^r$, one can ask whether there are other ways to make the $I^{(Nr)}$ bigger or the ideal I^r smaller and still always have containment.

For example, Craig Huneke raised the question: Given a reduced 0-dimensional subscheme $Z \subset \mathbb{P}^2$, to what extent is the result $I(4Z) \subseteq I(Z)^2$ optimal? In particular, is it always true that $I(3Z) \subset I(Z)^2$?

Experimentation and partial results suggested the answer is Yes (it is true for example if K has characteristic 2; see [2]). Thus I raised a more general conjecture [2], a simplified version of which is:

Conjecture 3.3.1. *Let $Z \subset \mathbb{P}^N$ be a fat point subscheme. Then $I((Nr - N + 1)Z) \subseteq I(Z)^r$ for all $r \geq 1$.*

Huneke's question is the case that $r = N = 2$. Over $\mathbb{C}$ this is the only case for which there are now counterexamples to the containment $I((Nr - N + 1)Z) \subseteq I(Z)^r$. Indeed, the first counterexample for any r and N over any field K was for $N = r = 2$ over $\mathbb{C}$: take the points Z of the Fermat arrangement for $n = 3$. Then $I(3Z) \not\subseteq I(Z)^2$ [11]. Additional counterexamples were soon found: there is a version in characteristic 3 [5], and additional positive characteristic counterexamples are now known for various r and N [16], and one can also take Z to be the points of the Fermat for any $n \geq 3$ [16], or the Klein or Wiman [3, 21].

Example 3.3.2. Here is Macaulay2 code for verifying $I^{(3)} \not\subseteq I^2$ for the $n^2 + 3$ points of the Fermat arrangement with $3n$ lines.

```
R=QQ[x,y,z];
n=5;
I=ideal(x^n-y^n, x^n-z^n);
J=ideal(x*y,x*z,y*z);
K=intersect(I,J);
K3=intersect(I^3,saturate(J^3));
isSubset(K3,K^2)
```

Example 3.3.3. Here is Macaulay2 code for verifying $I^{(3)} \not\subseteq I^2$ for the 49 points of the Klein arrangement of 21 lines.

```
-- Define the field
K=toField(QQ[c]/(c^2+c+2))
```

```
R=K[x,y,z];
-- Define the lines
F={x, x+c*y-z, -x+c*y-z, x+c*y+z, -x+y+c*z, y+z, c*x+y-z, z, c*x+y+z, c*x-y-z,
-x+z, -x-y+c*z, -x+y, c*x-y+z, -x+c*y+z, x+z, -y+z, x+y, x-y+c*z, x+y+c*z, y};
-- Find the product of the 21 linear forms
H=1_R;
apply(F,s->H=H*s);
-- Make a list of the ideals of the 49 intersection points of pairs of lines
W=subsets(21,2);
W4={}
apply(W,s->(flag=0;apply(W4,t->(if ideal(F_(t_0),F_(t_1))==ideal(F_(s_0),F_(s_1))
            then flag=1)); if flag==0 then W4=W4|{s}));
-- Define the ideal of the points
I=ideal(1_R);
apply(W4,s->(I=intersect(I,ideal(F_(s_0),F_(s_1)))));
-- Since H is in I^(3), it is enough to check that H is not in I^2
isSubset(ideal(H),I^2)
```

Example 3.3.4. Here is Macaulay2 code for verifying $I^{(3)} \not\subseteq I^2$ for the 201 points of the Wiman arrangement of 45 lines.

```
-- Define the field
K=toField(QQ[a]/(a^4-a^2+4))
R=K[x,y,z];
-- Define the lines
A=(-1/4)*(a^3-3*a-2);
B=(1/4)*(a^3+a-2);
F={y,(-1+A)*x+A*y+z,z,(1-A)*x+A*y-z,A*x+y+(-1+A)*z,-A*x+y+(1-A)*z,
(-1+A)*x-B*y+(-A-A*B)*z,(1-A)*x-B*y+(A+A*B)*z, (1-A)*x+A*y+z,
A*x+y+(1-A)*z, -x+(-1+A)*y+A*z, (-1-A*B)*x+y+(-1-B)*z,
(1-A)*x+B*y+(-A-A*B)*z, A*x+(B-A*B)*y+(-1-B)*z, (-A-A*B)*x+(1-A)*y-B*z,
(-1+A)*x+A*y-z, -A*x+y+(-1+A)*z, x+(-1+A)*y-A*z, (1+A*B)*x+y+(1+B)*z,
(-1+A)*x+B*y+(A+A*B)*z, -A*x+(B-A*B)*y+(1+B)*z, (A+A*B)*x+(1-A)*y+B*z,
(1+B)*x+(-1-A*B)*y+z, x+(-1+A)*y+A*z, x+(1-A)*y+A*z, (-1-A*B)*x+y+(1+B)*z,
(-A-B)*x+(-1+A+A*B)*y, -B*x+y+(-A+B-A*B)*z, (-1-A*B)*x-y+(1+B)*z,
(-1-B)*x+A*y+(B-A*B)*z, (-1-B)*x+(-1-A*B)*y-z, (A+B)*x+(-1+A+A*B)*y,
B*x+y+(A-B+A*B)*z, (1+B)*x+A*y+(-B+A*B)*z, (-1+A+A*B)*x+(-A-B)*z, x,
(-1-B)*x+A*y+(-B+A*B)*z, (-A-B)*y+(-1+A+A*B)*z, -B*x+y+(A-B+A*B)*z,
(1+B)*x-B*y+(1-A+B)*z, x-A*B*y+(1-A+B-A*B)*z, (-A-B)*y+(1-A-A*B)*z,
(1+B)*x+(1+A*B)*y-z, (A+B)*x+(1+B-A*B)*z, B*x+(-1+A-B)*y+(-1-B)*z};
-- Find the product of the 45 linear forms
H=1_R;
apply(F,s->H=H*s);
-- Make a list of the ideals of the 49 intersection points of pairs of lines
W=subsets(45);
W4={}
apply(W,s->(flag=0;apply(W4,t->(if ideal(F_(t_0),F_(t_1))==ideal(F_(s_0),F_(s_1))
            then flag=1)); if flag==0 then W4=W4|{s}));
-- Define the ideal of the points
I=ideal(1_R);
apply(W4,s->(I=intersect(I,ideal(F_(s_0),F_(s_1)))));
```

```
-- Since H is in I^(3), it is enough to check that H is not in I^2
isSubset(ideal(H),I^2)
```

Additional counterexamples arise by taking subsets of points of the Wiman arrangement.

Example 3.3.5. Here is Macaulay2 code for verifying $I^{(3)} \not\subseteq I^2$ for 200 of the 201 points of the Wiman arrangement of 45 lines. The missing point has multiplicity 3 in this case, but similar failures of containment occur by instead excluding a 4-point or a 5-point. The ideal of the 201 Wiman points is generated by three forms of degree 16. The ideal of the 200 points has an additional generator of degree 25, but the symbolic cube is generated in degree at most 49 (it has the usual degree 45 element, 20 generators of degree 48 and 6 of degree 49). Thus all homogeneous elements of I^2 of degree 49 or less vanish at all 201 points, but $I^{(3)}$ has elements of degree 49 that do not vanish at the missing point, and so $I^{(3)} \not\subseteq I^2$.

```
-- Define the field
K=toField(QQ[a]/(a^4-a^2+4))
R=K[x,y,z];
-- Define the lines
A=(-1/4)*(a^3-3*a-2);
B=(1/4)*(a^3+a-2);
F={y,(-1+A)*x+A*y+z,z,(1-A)*x+A*y-z,A*x+y+(-1+A)*z,-A*x+y+(1-A)*z,
(-1+A)*x-B*y+(-A-A*B)*z, (1-A)*x-B*y+(A+A*B)*z, (1-A)*x+A*y+z,
A*x+y+(1-A)*z, -x+(-1+A)*y+A*z, (-1-A*B)*x+y+(-1-B)*z,
(1-A)*x+B*y+(-A-A*B)*z, A*x+(B-A*B)*y+(-1-B)*z, (-A-A*B)*x+(1-A)*y-B*z,
(-1+A)*x+A*y-z, -A*x+y+(-1+A)*z, x+(-1+A)*y-A*z, (1+A*B)*x+y+(1+B)*z,
(-1+A)*x+B*y+(A+A*B)*z, -A*x+(B-A*B)*y+(1+B)*z, (A+A*B)*x+(1-A)*y+B*z,
(1+B)*x+(-1-A*B)*y+z, x+(-1+A)*y+A*z, x+(1-A)*y+A*z, (-1-A*B)*x+y+(1+B)*z,
(-A-B)*x+(-1+A+A*B)*y, -B*x+y+(-A+B-A*B)*z, (-1-A*B)*x-y+(1+B)*z,
(-1-B)*x+A*y+(B-A*B)*z, (-1-B)*x+(-1-A*B)*y-z, (A+B)*x+(-1+A+A*B)*y,
B*x+y+(A-B+A*B)*z, (1+B)*x+A*y+(-B+A*B)*z, (-1+A+A*B)*x+(-A-B)*z, x,
(-1-B)*x+A*y+(-B+A*B)*z, (-A-B)*y+(-1+A+A*B)*z, -B*x+y+(A-B+A*B)*z,
(1+B)*x-B*y+(1-A+B)*z, x-A*B*y+(1-A+B-A*B)*z, (-A-B)*y+(1-A-A*B)*z,
(1+B)*x+(1+A*B)*y-z, (A+B)*x+(1+B-A*B)*z, B*x+(-1+A-B)*y+(-1-B)*z};
-- Make a list of the ideals of the 49 intersection points of pairs of lines
W=subsets(45);
W4={}
apply(W,s->(flag=0;apply(W4,t->(if ideal(F_(t_0),F_(t_1))==ideal(F_(s_0),F_(s_1))
            then flag=1)); if flag==0 then W4=W4|{s}));
-- Find the multiplicity of the point where line i and line j intersect
W5={}
W5=apply(W4,s->(n=0;apply(F,t->(if isSubset(ideal(t),ideal(F_(s_0),F_(s_1)))
         then n=n+1)); W5|{s,n}));
toString W5
-- Here is what W5 turns out to be:
o15 = {{{0, 1}, 5}, {{0, 2}, 4}, {{1, 2}, 3}, {{2, 3}, 3}, {{0, 4}, 3}, {{1, 4},
      5}, {{2, 4}, 5}, {{3, 4}, 4}, {{1, 5}, 4}, {{2, 5}, 5}, {{3, 5}, 5}, {{0,
      6}, 3}, {{1, 6}, 4}, {{2, 6}, 5}, {{3, 6}, 5}, {{4, 6}, 3}, {{5, 6}, 4},
      {{1, 7}, 5}, {{2, 7}, 5}, {{3, 7}, 4}, {{4, 7}, 4}, {{5, 7}, 3}, {{0, 8},
      5}, {{5, 8}, 3}, {{6, 8}, 4}, {{7, 8}, 3}, {{0, 9}, 3}, {{1, 9}, 3}, {{6,
      9}, 3}, {{7, 9}, 3}, {{8, 9}, 4}, {{4, 10}, 4}, {{6, 10}, 5}, {{0, 11},
      3}, {{1, 11}, 3}, {{2, 11}, 3}, {{5, 11}, 5}, {{6, 11}, 3}, {{7, 11}, 3},
      {{8, 11}, 5}, {{9, 11}, 3}, {{10, 11}, 4}, {{0, 12}, 3}, {{1, 12}, 3},
```

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     {{3, 12}, 4}, {{4, 12}, 3}, {{5, 12}, 3}, {{7, 12}, 5}, {{8, 12}, 5}, {{9,
     12}, 3}, {{10, 12}, 3}, {{11, 12}, 5}, {{0, 13}, 5}, {{1, 13}, 3}, {{2,
     13}, 3}, {{3, 13}, 4}, {{5, 13}, 3}, {{6, 13}, 5}, {{7, 13}, 3}, {{8, 13},
     3}, {{9, 13}, 5}, {{11, 13}, 3}, {{12, 13}, 4}, {{0, 14}, 5}, {{1, 14},
     3}, {{3, 14}, 3}, {{4, 14}, 5}, {{5, 14}, 4}, {{6, 14}, 4}, {{8, 14}, 4},
     {{9, 14}, 4}, {{10, 14}, 5}, {{11, 14}, 3}, {{4, 15}, 3}, {{6, 15}, 3},
     {{7, 15}, 4}, {{12, 15}, 4}, {{13, 15}, 3}, {{3, 16}, 3}, {{7, 16}, 3},
     {{12, 16}, 4}, {{13, 16}, 4}, {{14, 16}, 3}, {{15, 16}, 4}, {{5, 17}, 4},
     {{7, 17}, 5}, {{11, 17}, 5}, {{14, 17}, 3}, {{2, 18}, 3}, {{3, 18}, 3},
     {{4, 18}, 5}, {{6, 18}, 3}, {{8, 18}, 3}, {{15, 18}, 5}, {{16, 18}, 3},
     {{1, 19}, 4}, {{3, 19}, 3}, {{4, 19}, 3}, {{5, 19}, 3}, {{8, 19}, 4}, {{9,
     19}, 4}, {{14, 19}, 3}, {{15, 19}, 5}, {{16, 19}, 3}, {{17, 19}, 3}, {{18,
     19}, 5}, {{1, 20}, 4}, {{2, 20}, 3}, {{3, 20}, 3}, {{4, 20}, 3}, {{6, 20},
     3}, {{8, 20}, 3}, {{9, 20}, 4}, {{15, 20}, 3}, {{16, 20}, 5}, {{18, 20},
     3}, {{1, 21}, 3}, {{3, 21}, 3}, {{4, 21}, 4}, {{5, 21}, 5}, {{7, 21}, 4},
     {{9, 21}, 3}, {{10, 21}, 3}, {{12, 21}, 3}, {{15, 21}, 4}, {{16, 21}, 4},
     {{0, 22}, 4}, {{2, 22}, 3}, {{10, 22}, 3}, {{14, 22}, 3}, {{7, 23}, 3},
     {{10, 23}, 3}, {{11, 23}, 4}, {{13, 23}, 3}, {{20, 23}, 5}, {{22, 23}, 3},
     {{6, 24}, 3}, {{13, 24}, 5}, {{17, 24}, 3}, {{18, 24}, 4}, {{20, 24}, 3},
     {{0, 25}, 3}, {{3, 25}, 5}, {{4, 25}, 3}, {{15, 25}, 3}, {{16, 25}, 5},
     {{17, 25}, 4}, {{18, 25}, 4}, {{19, 25}, 3}, {{0, 26}, 4}, {{2, 26}, 4},
     {{4, 26}, 3}, {{9, 26}, 3}, {{0, 27}, 3}, {{2, 27}, 4}, {{5, 27}, 3},
     {{10, 27}, 3}, {{22, 27}, 3}, {{23, 27}, 3}, {{1, 28}, 5}, {{8, 28}, 3},
     {{9, 28}, 5}, {{10, 28}, 4}, {{11, 28}, 4}, {{12, 28}, 3}, {{21, 28}, 3},
     {{15, 29}, 3}, {{24, 29}, 3}, {{2, 30}, 3}, {{17, 30}, 3}, {{24, 30}, 3},
     {{5, 31}, 3}, {{16, 31}, 3}, {{17, 32}, 3}, {{24, 32}, 3}, {{30, 32}, 3},
     {{8, 33}, 3}, {{23, 33}, 3}, {{0, 34}, 4}, {{2, 34}, 4}, {{26, 34}, 3},
     {{31, 34}, 3}, {{22, 35}, 3}, {{27, 35}, 3}, {{29, 35}, 3}, {{30, 35}, 3},
     {{32, 35}, 3}, {{33, 35}, 3}, {{1, 36}, 3}, {{10, 36}, 3}, {{17, 36}, 3},
     {{24, 36}, 3}, {{26, 37}, 3}, {{0, 38}, 3}, {{33, 38}, 3}, {{23, 39}, 3},
     {{29, 39}, 3}, {{3, 40}, 3}, {{10, 40}, 3}, {{17, 40}, 3}, {{31, 41}, 3}}
-- Remove one 3-point
W6 = {{0, 1}, {0, 2}, {2, 3}, {0, 4}, {1, 4}, {2, 4}, {3, 4}, {1, 5},
     {2, 5}, {3, 5}, {0, 6}, {1, 6}, {2, 6}, {3, 6}, {4, 6}, {5, 6}, {1, 7},
     {2, 7}, {3, 7}, {4, 7}, {5, 7}, {0, 8}, {5, 8}, {6, 8}, {7, 8}, {0, 9},
     {1, 9}, {6, 9}, {7, 9}, {8, 9}, {4, 10}, {6, 10}, {0, 11}, {1, 11}, {2,
     11}, {5, 11}, {6, 11}, {7, 11}, {8, 11}, {9, 11}, {10, 11}, {0, 12}, {1,
     12}, {3, 12}, {4, 12}, {5, 12}, {7, 12}, {8, 12}, {9, 12}, {10, 12}, {11,
     12}, {0, 13}, {1, 13}, {2, 13}, {3, 13}, {5, 13}, {6, 13}, {7, 13}, {8,
     13}, {9, 13}, {11, 13}, {12, 13}, {0, 14}, {1, 14}, {3, 14}, {4, 14}, {5,
     14}, {6, 14}, {8, 14}, {9, 14}, {10, 14}, {11, 14}, {4, 15}, {6, 15}, {7,
     15}, {12, 15}, {13, 15}, {3, 16}, {7, 16}, {12, 16}, {13, 16}, {14, 16},
     {15, 16}, {5, 17}, {7, 17}, {11, 17}, {14, 17}, {2, 18}, {3, 18}, {4, 18},
     {6, 18}, {8, 18}, {15, 18}, {16, 18}, {1, 19}, {3, 19}, {4, 19}, {5, 19},
     {8, 19}, {9, 19}, {14, 19}, {15, 19}, {16, 19}, {17, 19}, {18, 19}, {1,
     20}, {2, 20}, {3, 20}, {4, 20}, {6, 20}, {8, 20}, {9, 20}, {15, 20}, {16,
     20}, {18, 20}, {1, 21}, {3, 21}, {4, 21}, {5, 21}, {7, 21}, {9, 21}, {10,
     21}, {12, 21}, {15, 21}, {16, 21}, {0, 22}, {2, 22}, {10, 22}, {14, 22},
     {7, 23}, {10, 23}, {11, 23}, {13, 23}, {20, 23}, {22, 23}, {6, 24}, {13,
     24}, {17, 24}, {18, 24}, {20, 24}, {0, 25}, {3, 25}, {4, 25}, {15, 25},
```

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    {16, 25}, {17, 25}, {18, 25}, {19, 25}, {0, 26}, {2, 26}, {4, 26}, {9,
    26}, {0, 27}, {2, 27}, {5, 27}, {10, 27}, {22, 27}, {23, 27}, {1, 28}, {8,
    28}, {9, 28}, {10, 28}, {11, 28}, {12, 28}, {21, 28}, {15, 29}, {24, 29},
    {2, 30}, {17, 30}, {24, 30}, {5, 31}, {16, 31}, {17, 32}, {24, 32}, {30,
    32}, {8, 33}, {23, 33}, {0, 34}, {2, 34}, {26, 34}, {31, 34}, {22, 35},
    {27, 35}, {29, 35}, {30, 35}, {32, 35}, {33, 35}, {1, 36}, {10, 36}, {17,
    36}, {24, 36}, {26, 37}, {0, 38}, {33, 38}, {23, 39}, {29, 39}, {3, 40},
    {10, 40}, {17, 40}, {31, 41}}
-- Define the ideal of the points
I=ideal(1_R);
apply(W6,s->(I=intersect(I,ideal(F_(s_0),F_(s_1)))));
-- It turns out that the product H of the linear forms is in I^2 so we need to
-- compute I^(3), which is slow.
I3=ideal(1_R);
apply(W6,s->(I3=intersect(I3,(ideal(F_(s_0),F_(s_1)))^3)));
-- Alternatively, one could try: I3=saturate(I^3);
isSubset(I3,I^2)
```

Open Problem 3.3.6. *For which subsets Z of the 201 points of the Wiman arrangement do we have $I^{(3)} \not\subseteq I^2$, for $I = I(Z)$?*

Counterexamples also occur over the reals [8] and one of them can be made to work over the rationals [10]. This one is displayed in Figure 5. Take for Z the 19 crossing points of multiplicity 3. Then $I(3Z) \not\subseteq I(Z)^2$. In all of these counterexamples (i.e., the counterexample coming from the Fermat, Klein and Wiman line arrangements and the counterexample coming from the arrangement displayed in Figure 5), the failure is due to the fact that the form F coming from taking all of the lines of a line arrangement satisfies $F \in I(3Z)$ but $F \notin I(Z)^2$.

Another common feature of all of these counterexamples is that $t = \deg(F)/3$ is an integer, and the least m with $\dim[I(mZ)]_{mt} > 0$ is $m = 3$. One might hope that $I(3Z) \not\subseteq I(Z)^2$ if and only if $t = \deg(F)/3$ is an integer, and the least m with $\dim[I(mZ)]_{mt} > 0$ is $m = 3$. It is possible that this gives a necessary condition, but it is not sufficient.

For example, consider the line arrangement shown in Figure 5. It has 12 lines and 19 triple points. Let Z be the reduced scheme consisting of those 19 points. Then $I(3Z) \not\subseteq I(Z)^2$, 3 divides $\deg(F)$ and the least m with $\dim[I(mZ)]_{m4} > 0$ is $m = 3$. Now consider the line arrangement shown in Figure 6. It is the dual of a famous arrangement of 13 points, shown in Figure 7. Take as the lines of a line arrangement the 12 nondotted lines in Figure 6. It has 1 quadruple point, 18 triple points and 6 double points. Take for Z the 19 points of multiplicity more than 2. Then $I(3Z) \subseteq I(Z)^2$ even though 3 divides $\deg(F)$ and the least m with $\dim[I(mZ)]_{m4} > 0$ is $m = 3$.

This raises the question:

Open Problem 3.3.7. *Given a point set Z coming from a line arrangement such that $I(3Z) \not\subseteq I(Z)^2$, must there be a t such that the least m with $\dim[I(mZ)]_{mt} > 0$ is $m = 3$? Must $\deg(F)$ be a multiple of 3?*

All of the complex counterexamples have involved taking most of the points of a line arrangement of multiplicity at least 3, for line arrangements that only have a few or no simple crossings. A recent paper [1] leverages these counterexamples, by pulling them back by a finite cover of $\mathbb{P}^2$.

Example 3.3.8. Consider a line arrangement having $t_2 = 0$. Let F be the product of the linear forms of the lines. Let Z be the crossing points of the lines. Then since at least three lines cross at each crossing point, we have $F \in I(3Z)$. If it turns out that $F \notin I(Z)^2$, then we have $I(3Z) \not\subset I(Z)^2$. It can take work to check whether $F \notin I(Z)^2$ (see [21]). The simplest case might be as follows [5]. Take $\mathrm{char}(K) = 3$. Choose the point $p_0 = [0 : 0 : 1]$ (represented in Figure 8 by the

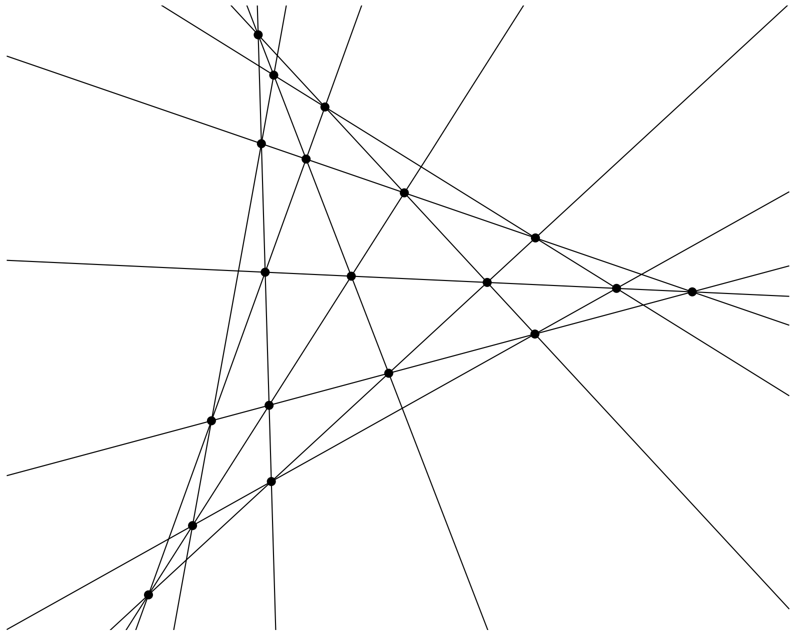

Figure 5. An arrangement of 12 lines with 19 triple points (and 9 double points).

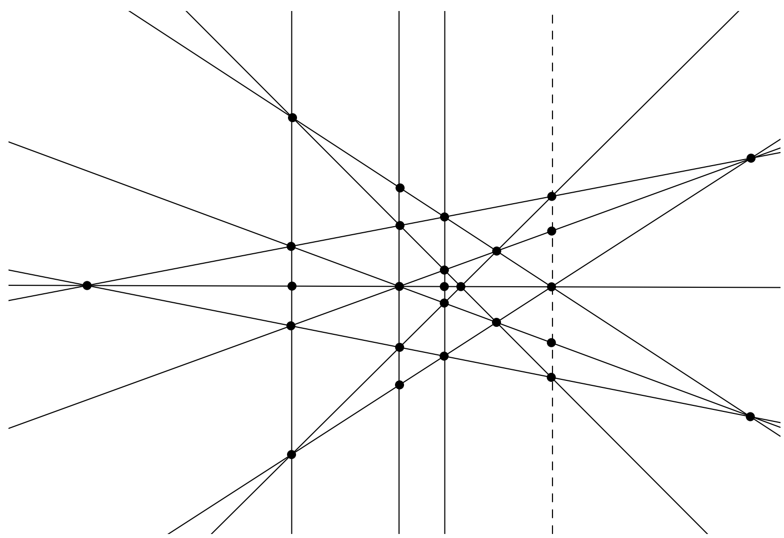

Figure 6. The dual of the McKee arrangement: 13 lines with 6 double points, 18 triple points and 3 quadruple points (one of which is at infinity, in the direction of the vertical lines).

open dot). There are 9 lines defined over the prime subfield which do not contain this point. They give an arrangement of 9 lines with 12 crossing points, and every crossing point has multiplicity 3. Take Z to be these 12 points. Note that for each point, the 3 lines of the arrangement through that point also go through 3 more of the points. By Bezout's Theorem, $\alpha(I(Z)) > 3$, and clearly $\alpha(I(Z)) \leq 4$.

The claim is that $\dim[I(Z)]_4 = 3$. There are various ways to verify this: for example, use facts about Hilbert functions, or run it on a computer. Here's a third way, in reference to Figure 8. Blow up the 11 points $p_2, \ldots, p_{12}$ shown to get a surface X. Let the blow up of E_i be the blow up of p_i. Denote the proper transforms of the lines L_i also by L_i. Then from

$$0 \to \mathcal{O}_X(L_4) \to \mathcal{O}_X(L_4 + L_3) \to \mathcal{O}_{L_3}(-1) \to 0$$

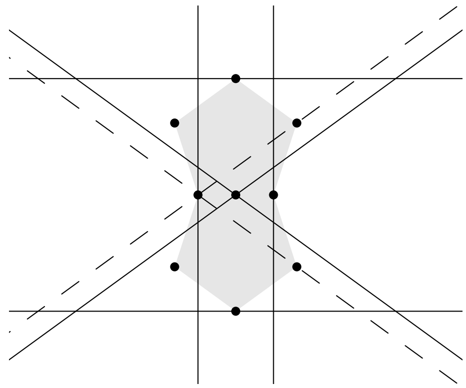

FIGURE 7. The McKee arrangement is based on two abutting regular pentagons. It has $n = 13$ points (including 4 at infinity in the directions of the lines) having only 6 ordinary lines (i.e., less than $n/2$ lines through exactly two points of the arrangement, shown as solid lines; note that the solid diagonal lines are parallel to the dashed lines).

we get $h^1(X, \mathcal{O}_X(L_4 + L_3)) = 0$. Now from

$$0 \to \mathcal{O}_X(L_4 + L_3) \to \mathcal{O}_X(L_4 + L_3 + L_2) \to \mathcal{O}_{L_2} \to 0$$

we get $h^1(X, \mathcal{O}_X(L_4 + L_3 + L_2)) = 0$. Note that $L_4 + L_3 + L_2 = 3L - E_5 - \cdots - E_{12}$. Now blow up p_1 to get Y. Then from

$$0 \to \mathcal{O}_Y(3L - E_5 - \cdots - E_{12}) \to \mathcal{O}_Y(4L - E_1 - \cdots - E_{12}) \to \mathcal{O}_{L_1} \to 0$$

we get $h^1(Y, \mathcal{O}_Y(4L - E_1 - \cdots - E_{12})) = 0$ and hence $h^0(Y, \mathcal{O}_Y(4L - E_1 - \cdots - E_{12})) = 3$, so $\dim[I(Z)]_4 = 3$. It's easy to check that the three quartics (namely $x^2y^2(x^2 - y^2)$, $x^2z^2(x^2 - z^2)$ and $y^2z^2(y^2 - z^2)$) given by the four F_3-lines through each of the coordinate vertices p_0, p_1, p_2 are linearly independent and so give a basis of $[I(Z)]_4$. Note that they all vanish at all 13 F_3-points of $\mathbb{P}^2$. Thus every element of $[I(Z)^2]_8$ vanishes at all 13 points. But $F(p_0) \neq 0$, so $F \notin [I(Z)^2]_8$. Hence $I(3Z) \not\subseteq I(Z)^2$.

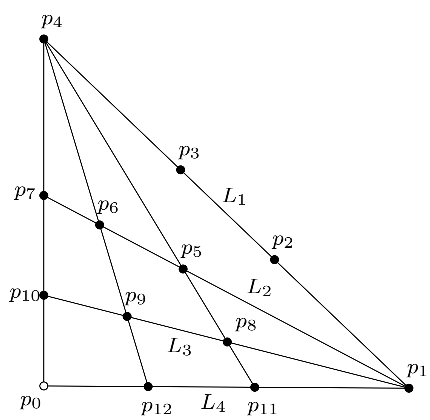

FIGURE 8. The 12 chosen points p_i of $\mathbb{P}^2$ over $K = F_3$.

Exercise 3.3.9. Let $Z \subset \mathbb{P}^2$ be the 12 points of the Fermat arrangement for $n = 3$. Let $I = I(Z)$. It is easy to check by computer that $\alpha(I) = \omega(I) = 4$ and we know from above that $I^{(3)} \not\subseteq I^2$. Show that $\widehat{\rho}(I) = \frac{4}{3} < \rho(I)$.

In fact, by [10, Theorem 2.1], we have $\widehat{\rho}(I) = \frac{n+1}{n}$ for the ideal I of the n^2+3 points of the Fermat arrangement for $n \geq 3$, and $\rho(I) = \frac{3}{2}$.

If there are complex line arrangements in addition to the Fermat, Klein and Wiman with $t_2 = 0$, it seems reasonable to expect they would give additional counterexamples.

Another way to address optimality of $I(rNZ) \subset I(Z)^r$ is to make the right hand side of the containment smaller. This led to the following conjecture [15]:

Conjecture 3.3.10. *Let $Z \subset \mathbb{P}^N$ be a fat point subscheme and let $M = R_1R$ for $R = K[\mathbb{P}^N]$. Then $I(NrZ) \subseteq M^{Nr-r}I(Z)^r$ for all $r \geq 1$.*

So far this conjecture remains open in all characteristics. The motivation was a conjecture of Chudnovsky [4], aimed at improving the bound of Waldschmidt and Skoda (see Exercise 1.3.3(f)):

Conjecture 3.3.11. *Let $Z \subset \mathbb{P}^N$ be a fat point subscheme (in the original statement, Z was reduced). Then*

$$\frac{\alpha(I(Z)) + N - 1}{N} \leq \widehat{\alpha}(I(Z)).$$

Exercise 3.3.12. Show that Conjecture 3.3.10 implies Conjecture 3.3.11. (Hint: Mimic Exercise 1.3.3(f).)

Likewise, if $I((N+n-1)rZ) \subseteq M^{(N-1)r}I(nZ)^r$ were true, then we would get an affirmative answer to a question posed in [15]; i.e., we would have

$$\frac{\alpha(I(nZ)) + N - 1}{N + n - 1} \leq \widehat{\alpha}(I(Z)).$$

When $N = 2, n = 1$ and Z is reduced this is a result of Chudnovksy [4]; see [15] for one proof. Here's a more geometric proof (this was probably along the lines of how Chudnovksy did it, but he wasn't very explicit in his paper). Let $I = I(Z)$. Pick the largest subset $S \subseteq Z$ (which need not be unique) of the points of Z imposing independent conditions on forms of degree $a = \alpha(I)$. Let $s = |S|$. Thus $\dim[I]_a = \binom{a+2}{2} - s$. If $s < \binom{a+1}{2}$, then $\dim[I]_{a-1} > 0$. If $s \geq \binom{a+2}{2}$, then $\dim[I]_a = 0$. Thus $\binom{a+1}{2} \leq s < \binom{a+2}{2}$. So we can pick a subset $U \subseteq S$ such that $|U| = \binom{a+1}{2}$. Note that $\alpha(I(U)) = \operatorname{reg}(I(U))$ by Definition 3.2.2, so $I(U)$ is generated in degree at most $a = \alpha(I(U))$ by Fact 3.2.3. In particular, $I(U)_a$ has 0-dimensional zero locus, thus given a nonzero $F \in I(mZ)_{a_m}$ of degree $a_m = \alpha(I(mZ))$, we can pick a nonzero $G \in I(U)_a$ with no components in common with F, hence, by Bezout's Theorem, we have $a_m a = \deg(F)\deg(G) \geq m|U| = m\binom{a+1}{2}$; i.e., $a_m a \geq m\binom{a+1}{2}$ and hence $\frac{a_m}{m} \geq \frac{a+1}{2}$, so $\widehat{\alpha}(I(Z)) \geq \frac{\alpha(I(Z))+1}{2}$.

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4. A new perspective on the SHGH Conjecture

4.1. Conditions imposed by fat points.

Open Problem 4.1.1. *Find all degrees t and all fat point schemes $X = \sum_i m_i p_i \subset \mathbb{P}^2$ where the points p_i are general such that X unexpectedly fails to impose independent conditions on $V = (K[\mathbb{P}^2])_t$; i.e.,*

$$\dim I(X)_t > \min\left(0, \dim V - \sum_i \left(\binom{m_i+1}{2}\right)\right).$$

The SHGH Conjecture gives a conjectural solution for this.

Exercise 4.1.2. Suppose we are given a smooth rational surface X, an exceptional curve E (i.e., a smooth rational curve with $E^2 = -1$), and a divisor F on X. If $h^0(X, \mathcal{O}_X(F)) > 0$, then $h^1(X, \mathcal{O}_X(F + rE)) > 0$ if $(F + rE) \cdot E \leq -2$.

If F is a divisor on a surface S obtained by blowing up general points p_i of $\mathbb{P}^2$, the SHGH Conjecture says:

Conjecture 4.1.3. *If $h^0(S, \mathcal{O}_S(F)) > 0$ and $h^1(S, \mathcal{O}_S(F)) > 0$, then there is an exceptional curve E on S such that $F \cdot E \leq -2$.*

If this is true, then standard techniques allow one to compute $h^0(S, \mathcal{O}_S(F))$ exactly for any divisor F on S. For simplicity let's assume $s > 1$.

Here is the idea: Given $F = tL - \sum_i m_i E_i$ for general points p_i, there is an algorithmic procedure which gives either a nef divisor H such that $H \cdot F < 0$ (and hence $h^0(S, \mathcal{O}_S(F)) = 0$), or which gives a Zariski-like decomposition $F = A + \sum_i c_i C_i$ such that $A \cdot E \geq 0$ for all exceptional curves E, and the C_i are exceptional with $c_i \geq 0$, $A \cdot C_i = 0$ and $C_i \cdot C_j = 0$ for all $i \neq j$.

In this case $h^0(S, \mathcal{O}_S(F)) = h^0(S, \mathcal{O}_S(A)) \geq \frac{A^2 - K_S A}{2} + 1$; the content of the SHGH Conjecture is that this is an equality.

Exercise 4.1.4. Assuming the SHGH Conjecture, if $C^2 < 0$ for a reduced irreducible curve C on a blow up X of $\mathbb{P}^2$ at general points p_i, show that C is an exceptional curve.

A sample more general problem: Find examples of reduced point schemes $Z \subset \mathbb{P}^2$ and t such that $\dim I(Z)_t \leq 3$ but for every p there is a curve of degree t containing Z singular at p.

We will relate this to the following open problem (this and all that follows is based on [1], which in turn was motivated by the paper [2]):

Open Problem 4.1.5. *Find all t and fat points $Z = \sum_j a_j q_j$ and general fat points $X = \sum_i m_i p_i$ in $\mathbf{P}^2$ such that X unexpectedly fails to impose independent conditions on $V = I(Z)_t$; i.e.,*

$$\dim(I(X)_t \cap V) > \min\left(0, \dim V - \sum_i \binom{m_i+1}{2}\right). \tag{1}$$

Example 4.1.6. Each of the following give examples of an X and Z where X unexpectedly fails to impose independent conditions on $V = I(Z)_t$.

(a) If $Z = 0$, this is just is just a case of Problem 4.1.1, so is solved by the SHGH Conjecture.
(b) If Z consists of fat points where the points are general, this also in principle is solved by the SHGH Conjecture.
(c) If Z is reduced and consists of the the 7 points of the Fano plane (so $\mathrm{char}(K) = 2$), then this is an example of both the sample problem and Problem 4.1.5, where we have $V = I(Z)_t$, $t = 3$, $X = 2p$, $\dim V = 3$. Being singular at p imposes 3 conditions, so we expect no curve, but for every point p there is a cubic through Z singular at p (specifically

$F = \alpha^2 yz(y+z) + \beta^2 xz(x+z) + \gamma^2 xy(x+y)$ vanishes at the 7 points and is singular at $p = (\alpha, \beta, \gamma)$).

(d) Take $X = mp$ and let Z be reduced, consisting of the points dual to the $3n$ Fermat lines where $n \geq 5$, $n+1 \leq m \leq 2n-4$ and $t = m+1$. (Its splitting type, defined below, is $(n+1, 2n-2)$.)

(e) Take $X = mp$ and let Z be reduced, consisting of the points dual to the Klein lines; $m = 9$ and $t = 10$. (Its splitting type is $(9, 11)$.)

(f) Take $X = mp$ and let Z be reduced, consisting of the points dual to the Wiman lines; $19 \leq m \leq 23$ and $t = m+1$. (Its splitting type is $(19, 25)$.)

It is not obvious how to find such examples. Doing so uses some theory. Let $Z = q_1 + \cdots + q_r$ be a reduced point scheme in $\mathbf{P}^2$, ℓ_j the linear form dual to q_j, and let $F = \ell_1 \cdots \ell_r$. Now assume that char(K) does not divide deg(F), and let $\mathcal{J}_F$ be the Jacobian sheaf (i.e., the sheafification of the ideal (F_x, F_y, F_z) generated by the partial derivatives of F), and $\mathcal{D}$ the syzygy bundle; i.e., the sheaf defined by the exact sheaf sequence

$$0 \to \mathcal{D} \to \mathcal{O}^3 \to \mathcal{J}_F(r-1) \to 0.$$

Restricted to a general line L we get $\mathcal{D}|_L = \mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b)$ where $a + b = d - 1$, $a \leq b$. Call (a, b) the splitting type of Z.

To state the results, it's convenient to introduce a quantity t_Z, defined as the least j such that $\dim I(Z)_{j+1} > \left(\binom{j+1}{2}\right)$.

Theorem 4.1.7. *Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$ of splitting type (a, b) with $X = mp$ for a general point $p \in \mathbb{P}^2$. Then (1) holds for some degree t if and only if $a_Z < t_Z$. Furthermore, in this case the degrees t for which (1) holds are precisely those in the range $a_Z < t < b_Z$.*

Here's another version:

Theorem 4.1.8. *Given a reduced 0-dimensional subscheme $Z \subset \mathbb{P}^2$ of splitting type (a, b) and $X = mp$ for a general point $p \in \mathbb{P}^2$, then (1) holds in degree $t = m+1$ if and only if*

(a) *$a \leq m \leq b - 2$ and*

(b) *$t_Z + 1 \geq reg(I(Z))$.*

Proof. See [1]. The proof uses ideas of [3] to relate syzygies and singular curves via the point line correspondence in the plane. □

Here's an example run using Macaulay2 [4], version 1.4.

Example 4.1.9. Let's verify an instance of Theorem 4.1.7. Consider the Fermat line arrangement for $n = 5$. The form defining the lines is $F = (x^5 - y^5)(x^5 - z^5)(y^5 - z^5)$. The scheme Z of points dual to the 15 lines has ideal $I(Z) = (x^5 + y^5 + z^5, xyz)$. First we compute te splitting type. In this case we do not need to restrict to a general line, since the syzygy bundle is free, so we can read the splitting off directly from the first syzygy module in a minimal free resolution of the Jacobian ideal $J = (F_x, F_y, F_z)$.

```
i1 : R=QQ[x,y,z];

i2 : F=(x^5-y^5)*(x^5-z^5)*(y^5-z^5);

i3 : J=ideal(jacobian(ideal(F)));

i4 : betti res J
```

```
             0 1 2
o4 = total: 1 3 2
         0: 1 . .
         1: . . .
         .
         .
         .
        11: . . .
        12: . . .
        13: . 3 .
        14: . . .
        15: . . .
        16: . . .
        17: . . .
        18: . . 1
        19: . . .
        20: . . 1
```

We see that J has three generators of degree 14, as expected, and the generators of the syzygy module have degrees 6 and 8, giving the splitting type $(6, 8)$, in agreement with Example 4.1.6(d).

It is possible to pick a generic point for p. We can take $p = (A, B, 1)$ where A ad B are variables, but this requires working over the field $K = \mathbb{Q}(A, B)$. The Macaulay2 commands for this are $K = \mathrm{frac}(\mathbb{Q}[A, B])$, $R = K[x, y, z]$. But working over $K = \mathbb{Q}(A, B)$ is hard for my laptop running Macaulay2, so we pick a random point for p instead.

```
i1 : R=QQ[x,y,z];

i2 : p=ideal(random(1,R), random(1,R));

i3 : Z=ideal(x^5+y^5+z^5,x*y*z);

i4 : I5=intersect(p^5,Z);

i5 : betti res I5

             0 1 2
o5 = total: 1 7 6
         0: 1 . .
         1: . . .
         2: . . .
         3: . . .
         4: . . .
         5: . . .
         6: . 6 4
         7: . 1 2
-- Note I5 has no element of degree 6.

i6 : I6=intersect(p^6,Z);

i7 : betti res I6
```

```
              0 1 2
o7 = total: 1 7 6
          0: 1 . .
          1: . . .
          2: . . .
          3: . . .
          4: . . .
          5: . . .
          6: . 1 .
          7: . 6 5
          8: . . 1
-- So the least m for which I(Z+mp) has an element of degree m+1 is m=6;
-- i.e., a_Z = 6.

i8 : for i from 2 to 8 do print {i,hilbertFunction(i,R)-hilbertFunction(i,I6),
     hilbertFunction(i,R)-hilbertFunction(i,Z)}
{2, 0, 0}
{3, 0, 1}
{4, 0, 3}
{5, 0, 7}
{6, 0, 13}
{7, 1, 21}
{8, 9, 30}

-- 6p should impose 21 conditions on I(Z)_7, making I(Z+6p)_7 = 0
-- but instead we see dim I(Z+6p)_7 = 1, so 6p unexpectedly
-- fails to impose independent conditions on I(Z)_7.

i9 : I7=intersect(p^7,Z);

i10 : for i from 2 to 9 do print {i,hilbertFunction(i,R)-hilbertFunction(i,I7),
      hilbertFunction(i,R)-hilbertFunction(i,Z)}
{2, 0, 0}
{3, 0, 1}
{4, 0, 3}
{5, 0, 7}
{6, 0, 13}
{7, 0, 21}
{8, 2, 30}
{9, 12, 40}

-- 7p should impose 28 conditions on I(Z)_8, making I(Z+6p)_8 = 2
-- and it is, so 7p imposes independent conditions I(Z)_8.
```

Exercise 4.1.10. Let Z be the 9 points of shown in Figure 2. Let p be a general point. Show that there is an irreducible quartic though the points of Z, with a triple point at p. Conclude that it is unique. (Hint: Assume the four general points in the figure, shown in black, are $(001), (010), (100), (111)$. The other 5 points then become $(011), (101), (110), (1-10), (112)$. Use a computer to compute the splitting type, and t_Z, and apply Theorem 4.1.7. Then use Bezout to show irreducibility and uniqueness.)

4.2. **Curves and syzygies.** This result raises the question why there should be a connection between curves and syzygies. Assume $\mathrm{char}(K) = 0$. Let $s = (s_0, s_1, s_2)$ be a minimal syzygy (i.e., of least degree possible) of $\nabla F = (F_x, F_y, F_z)$; i.e., $s \cdot \nabla F = 0$, meaning

$$s_0 F_x + s_1 F_y + s_2 F_z = 0.$$

Since s is minimal, the s_i have no nonconstant common factor. Thus s defines a rational map $s : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined at all but a finite set of points. Therefore, s restricts as a morphism $s|_L : L \to \mathbb{P}^2$ to a general line L.

Exercise 4.2.1. If $s : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined at $p \in \mathbb{P}^2$ but $s(p) = p$, show that $F(p) = 0$. (Hint: Look at $s \cdot \nabla F$ and use Euler's identity $xF_x + yF_y + zF_z = \deg(F)F$.)

Exercise 4.2.2. Let $f = (f_0, f_1, f_2) = (x, y, z) \times s$, so $f_0 = ys_2 - zs_1$, $f_1 = -(xs_2 - zs_0)$ and $f_2 = xs_1 - ys_0$. Let $\ell = Ax + By + Cz$ be a linear factor of F. For any point $p = (a, b, c)$ for which $s(p) \neq p$, show that $f(p)$ is the point dual to the line through p and $s(p)$. If in addition $\ell(p) = 0$ but p is not a singular point of $F = 0$, show that $\ell(s(p)) = 0$ and conclude that $f(p)$ is the point dual to the line defined by ℓ. (Hint: apply the product rule for ∇F.)

Exercise 4.2.3. If s is not the identity on any line defined by $F = 0$, show that $f|_L :\to \mathbb{P}^2$ defines a morphism whose image contains the points Z dual to the lines defined by the linear factors of F and such that the points of $L \cap s(L)$ map to the point dual to L. (Aside: In fact, $s(L)$ is a curve of degree $\deg(s_i) + 1$ that contains Z and has a point of multiplicity $\deg(s_i)$ at the point dual to L.)

Exercise 4.2.4. If s is not the identity on any line defined by $F = 0$, show that $(x, y, z) \times f = -\deg(F)Fs$, hence we can recover s from f.

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